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## Combinatorial Ordering
## and
## the Geometric Embedding of Graphs

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ii

COMBINATORIAL ORDERING
AND THE GEOMETRIC EMBEDDING OF GRAPHS[*]

## ABSTRACT

This thesis introduces a new graph-theoretic structure − the (2, 1)-connected
sequence − with direct applicability to the embedding of both planar and non-
planar graphs. It is proven that: (1) the nodes of a graph can be ordered so
as to form a (2, 1)-connected sequence, regardless of whether the graph is
planar or nonplanar, and (2) such a sequence yields a new and exceptionally
simple technique for planarity testing and embedding. All algorithms are
proven to operate within a time bound proportional to the square of the number
of nodes or edges in the graph.

## ACKNOWLEDGEMENTS

iv

# TABLE OF CONTENTS

# LIST OF DEFINITIONS, LEMMAS

## AND THEOREMS

### (in order of appearance)

Chapter 1

## INTRODUCTION

### Section 1.  PERSPECTIVE

The study of the geometric embedding of graphs is an active
subject of both theoretical and practical interest (see [Owens69]).
Much of the research in this area of graph theory has been directed
toward the unresolved problem of finding methods for embedding or
"drawing" a nonplanar graph in one or more planes in such a way
as to satisfy some chosen criterion, such as minimality of the
number of overlaps or planes.  Despite much effort, few conclusive
results have been established.

Embedding problems can be viewed as a generalization of the
problem of finding algorithms for testing whether a graph is planar
and for producing a planar embedding when one exists.   Several
algorithms for planarity testing and embedding now exist in the
literature, for example, that of A. J. Goldstein [Gol63] and the
well-known algorithm of Auslander and Parter [AusP61].

Unfortunately, neither of the algorithms described by these
authors, nor any other known planarity algorithm, gives any substantial
clue as to how to extend the algorithm in order to attack embedding

problems of nonplanar graphs. In comparison, this thesis introduces a new graph-theoretic structure — the (2, 1)-connected sequence — with direct applicability to the embedding of both planar and nonplanar graphs. It is proven that: (1) the nodes of a graph can be ordered so as to form a (2, 1)-connected sequence, regardless of whether the graph is planar or nonplanar, and (2) such a sequence yields a new and exceptionally simple technique for planarity testing and embedding. While the scope of the thesis does not include explicit exploration of embedding problems of nonplanar graphs, the existence and properties of (2, 1)-connected sequences suggest substantial applications to such problems; it is anticipated that these applications will be pursued in the future (see Chapter 4, Section 2).

A noteworthy feature of the planarity technique mentioned in (2) is that the embedding of a planar graph is drawn in successive steps "from the interior outward" — that is, in such a way that nodes and edges are successively placed outside the periphery of the set of edges and nodes that have already been drawn (see Section 2, Figure 1.2). No other existing planarity algorithm possesses this feature, which makes the construction of a planar embedding extremely simple.

All the algorithms presented in the thesis are efficient, in the sense that they run within a time bound that is proportional to $m^2$ (or equivalently — insofar as planarity testing and embedding are concerned — proportional to $n^2$), where $m$ and $n$ are the number of edges and nodes, respectively (see Chapter 4). Moreover, it may be possible to prove that these algorithms actually run within a strictly smaller time bound.

The sequel focuses on 3-connected graphs (Definition 2.1.11, Chapter 2). By a well-known result of Saunders MacLane regarding the decomposition of graphs into 3-connected pieces (see [MacL37b]), the assumption of 3-connectivity is not a restriction on the applicability of the present work.

Section 2.  BACKGROUND


This section describes the basic ideas underlying the formal con-cepts and algorithms presented in succeeding chapters.

### 2.1  Concepts

In order to keep the discussion as concrete as possible, let us consider a specific graph — for example, the graph $\overline{G}$ consisting of the nodes $\overline{V} = \{A, B, \ldots, I\}$ and the following set of edges $\overline{E}$:

| | | | |
|---|---|---|---|
| AC | BF | CI | EI |
| AD | BG | DI | FG |
| AI | CE | DF | FH |
| BC | CH | EH | GH . |

$\overline{G}$ is a planar graph, as shown by the planar embedding in Figure 1.1.
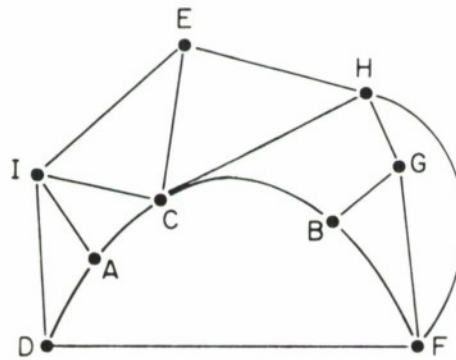


Figure 1.1   A Planar Embedding

The above figure can be constructed by means of the sequence
of steps (1)-(5) shown in Figure 1.2.
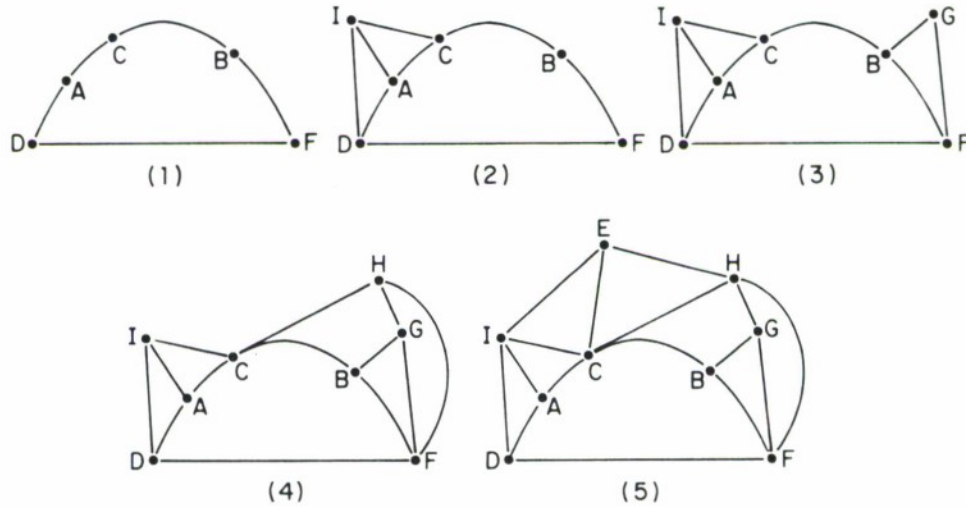


Figure 1.2   Construction of Planar Embedding in Figure 1.1

We can summarize steps (1)-(5) as follows:  first draw circuit D, F, B, C,
A, D; then "expand" this circuit by adding nodes I, G, H, E,  in that
order.  The essential part of this "summary" is the sequence

$$\langle D, F, B, C, A, I, G, H, E \rangle,$$

which tells us in what succession to add the nodes in order to repro-
duce the steps in Figure 1.2.

To anyone interested in  geometric  embedding  problems,
the above observations raise a tantalizing question.  Since the sequence
$\langle D, F, B, C, A, I, G, H, E \rangle$ tells us essentially how to construct a planar
embedding of $\overline{G}$,  is there a way to extract this useful sequence directly

from the abstract specification of $\overline{G}$? Specifically, do sequences such as $\langle D, F, B, C, A, I, G, H, E \rangle$ possess some simple, combinatorial property that assures their effectiveness as "embedding-generating" sequences, and can sequences with this property be efficiently constructed? The succeeding chapters are devoted to stating this question in precise terms and to answering it affirmatively.

The most remarkable result is that such a combinatorial property not only exists, but in fact is independent of the planarity or non-planarity of the graph. As a consequence, possibilities immediately arise for applications to the problem of layout for nonplanar graphs. While such applications are beyond the scope of this thesis, several comments on the subject are offered at the end of Chapter 4.

The central concept, then, is the combinatorial property just discussed. By way of motivating the subsequent formulation of this property, let us return to graph $\overline{G}$. Observe that the construction in Figure 1.2 partitions the edges of $\overline{G}$ into a sequence of five sets, corresponding to the five steps in the construction; these sets are illustrated in Figure 1.3.
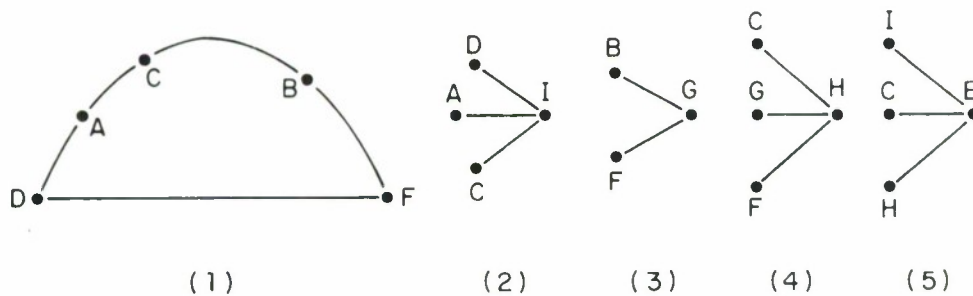


Figure 1.3  Circuit-Star Decomposition

Note that the first of these sets is a circuit, and every other set is a "star," i.e., a set of edges emanating from a single node, the "center." Notice also that the "center" of each star (other than the last) is adjacent to a node in a succeeding star. We shall use the term "circuit-star decomposition" informally, to refer to an edge-decomposition with the above characteristics.

The desired combinatorial property is formulated as the concept of "(2, 1)-connectedness," presented in Definition 2.2.1, Chapter 2. This concept is essentially a formalization of the sequence of "centers" of a "circuit-star decomposition"; the formalization is expressed succinctly in terms of degree of connectivity (see Definition 2.1.11), without reference to circuits or "stars."

## 2.2 Algorithms

Apart from showing that "(2, 1)-connected" sequences possess the desired "embedding-generating" character, the following chapters are devoted primarily to proving that (2, 1)-connected sequences exist and can be efficiently constructed for any 3-connected graph, whether planar or nonplanar. The chief tool in this construction is the "chain sequence" (Definition 2.2.6, Chapter 2), which is a sequence of paths satisfying special connectivity requirements. The relevance of a chain sequence is that, if $\langle F^0, F^1, \ldots, F^q \rangle$ is a "complete" chain sequence (i.e., a chain sequence containing <u>all</u> nodes of a graph G) and if $\hat{F}^i$ denotes the sequence of nodes in the path $F^i$, minus the first and last nodes, then (ignoring secondary details) $F^0 \circ \hat{F}^1 \circ \ldots \circ \hat{F}^q$ is a (2, 1)-connected sequence for G (where "$\circ$" denotes concatenation). Chain sequences have the advantage of being amenable to an iterative method

of construction.

In view of the importance of "complete" chain sequences, it is worthwhile to sketch the main features of the technique used to generate them. Initially, a sequence of paths $r = \langle R^1, R^2 \rangle$ is constructed (for details, see Chapter 2, Section 2.2). If r does not contain all the nodes of the graph G, then the sequence r is "augmented," i.e., subjected to a structure-preserving operation which enlarges the set of nodes belonging to paths of r. Essentially, this augmentation is accomplished by replacing a path S in r with two paths $S_1, S_2$ constructed by means of one of three methods illustrated in Figure 1.4; the nodes of path Q (see Figure 1.4) other than its endpoints, do not belong to any paths in r.
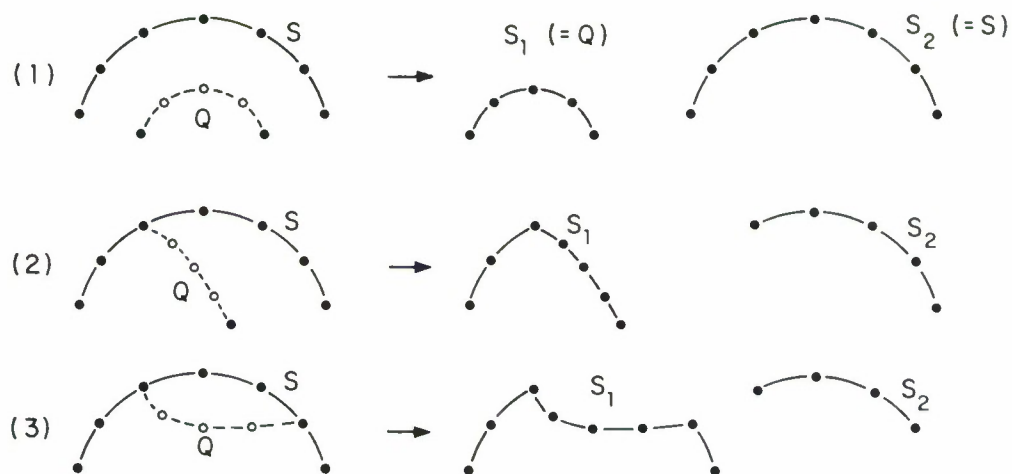
Figure 1.4   Augmentation Method

The critical issue is to show that the augmentation is structure-preserving (in a formal sense that is defined in Chapter 2), and this issue reduces, intuitively speaking, to proving that a "judicious" choice of the "augmenting" path Q can be made.

In summary, the succeeding chapters establish the following results:

(1)    For any 3-connected graph G (whether planar or non-planar), there exists a (2, 1)-connected ordering of the nodes of G and a procedure for generating such an ordering.

(2)    If G is planar, then a planar embedding of G can be generated from any (2, 1)-connected sequence via the "expansion algorithm" (a formal procedure corresponding to the method illustrated in Figure 1.2 above).

(3)    The above procedures are computationally efficient; namely, they run within a time bound that is proportional to $m^2$ (or equivalently — insofar as planarity testing and embedding are concerned — proportional to $n^2$), where m and n are the number of edges and nodes, respectively.

Thus, the procedure for constructing (2, 1)-connected sequences, when coupled with the "expansion algorithm," provides an efficient planarity-testing and planar-embedding technique. Of particular significance, moreover, is the fact that the procedure for constructing (2, 1)-connected sequences applies to nonplanar as well as planar graphs, and thereby opens the way to future exploration of the layout problem by means of (2, 1)-connected sequences and related techniques.

## Section 3.  PLAN OF PRESENTATION

Chapter 2 presents the concepts and algorithms discussed above. Chapter 3 establishes the validity of these algorithms.  For the convenience of the reader, these two chapters are organized in parallel fashion; for example, Section 2.2 of Chapter 2 specifies the algorithm for constructing $(2, 1)$-connected sequences,  while Section 2.2 of Chapter 3 contains the proof of the validity of this algorithm.  In addition, Sections 2.1 and 3.1 of Chapter 2 provide an informal overview of the entire presentation; these two sections,  in conjunction with Chapter 1, can be read as an intuitive digest of the whole thesis.

Chapter 4 establishes a bound on computational complexity, discusses future research, and closes with a brief summary.

Chapter 2

ALGORITHMS

Section 1. FOUNDATIONS

## 1.1 Perspective

This chapter gives a detailed presentation of the algorithms introduced in Chapter 1. Section 1 contains standard graph-theoretic definitions; Section 2 describes the construction of (2, 1)-connected sequences; Section 3 presents the expansion algorithm. Sections 2 and 3 each begin with a subsection devoted to definitions and an intuitive overview of succeeding details.

## 1.2 Standard Definitions (see [Liu68])

Definition 2.1.1 A (finite, abstract) graph $G = (V, E, P)$ is a triplet consisting of a finite set V (the nodes of G, denoted by "nodes (G)"), a finite set E (the edges of G denoted by "edges (G)"), and a relation $P \subseteq V \times E$ (where $\langle v, e \rangle \in P$ is read "v is an endpoint of e") such that, for each edge e, there exists at least one and at most two nodes which are endpoints of e.

If $e \in E$ has exactly one endpoint, then e is called a self-loop. If endpoints $(e_1)$ = endpoints $(e_2)$ and $e_1 \neq e_2$, then $e_1$ and $e_2$ are said to be parallel edges.

Convention 2.1.2 Throughout the sequel, it is to be understood that the term graph refers to a graph without self-loops and parallel edges. For

such graphs, the set E can be specified by a set of unordered pairs of nodes, and explicit mention of P is unnecessary. In this case we speak of the graph $G = (V, E)$. An edge with endpoints $v, w$ will be denoted by $(v, w)$. A <u>subgraph</u> of $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subset V$ and $E' \subseteq E$.

<u>Definition 2.1.3</u>  Two nodes $v_1, v_2$ in a graph $G = (V, E)$ are said to be <u>adjacent</u> (relative to G) if $(v_1, v_2) \in E$. The <u>degree</u> of a node $v$ (relative to G) is the number of nodes which are adjacent to $v$ (relative to G). Two sets of nodes $H_1, H_2$ in G are said to be <u>adjacent</u> if there exists adjacent nodes $v_1 \in H_1$, $v_2 \in H_2$.

<u>Definition 2.1.4</u>  A <u>path</u> in a graph G is a sequence of nodes $\langle v_1, \ldots, v_t \rangle$ $(t \geq 1)$ such that $v_i, v_{i+1}$ are adjacent $(i \geq 1)$. The <u>edges</u> of the path $\langle v_1, \ldots, v_t \rangle$ are the edges $(v_i, v_{i+1})$ $(i = 1, \ldots, t-1)$. The <u>length</u> of the path is $t-1$ (the number of edges). If $t = 1$ (i.e., the path is of length 0), then the path is said to be <u>empty</u>. The <u>interior</u> of a path $\langle v_1, \ldots, v_t \rangle$ (denoted $\text{int} \langle v_1, \ldots, v_t \rangle$) is the set of nodes $\{ v_2, \ldots, v_{t-1} \}$. The <u>endpoints</u> of $\langle v_1, \ldots, v_t \rangle$ are $v_1, v_t$. A <u>subpath</u> of a path P is a sequence of consecutive nodes of P; i.e., "P' is a subpath of P" implies that the nodes of P' appear in the same order as they do in P.

<u>Definition 2.1.5</u>  A path is said to be <u>simple</u> if all its edges are distinct, and <u>elementary</u> if all its nodes are distinct.

<u>Definition 2.1.6</u>  A circuit is a path $\langle v_1, \ldots, v_t \rangle$ of length $\geq 3$, such that $v_1 = v_t$. A circuit is said to be <u>simple</u> if all its edges are distinct, and <u>elementary</u> if the nodes $v_1, \ldots, v_{t-1}$ are all distinct.

Convention 2.1.7   In dealing with paths and circuits, we will occasionally identify the path or circuit with the graph composed of its nodes and edges.  The set of nodes in a path P will be denoted by "nodes(P)."

Definition 2.1.8   A graph G is said to be connected if for every two nodes $v_1, v_2$ there is a path in G with endpoints $v_1, v_2$; otherwise, G is said to be disconnected.

Definition 2.1.9   Suppose G = (V, E) and $V' \subseteq V$.  The section graph of $V'$ in G (denoted by "G[V']") is the subgraph of G composed of the nodes $V'$ and all edges of G having both endpoints in $V'$.

Definition 2.1.10   A set of nodes S in a graph G = (V, E) is called a k-separating set if $|S| \leq k$ and the section graph of V–S (in G) is disconnected.  A separating set is any set which is a k-separating set, for some k.

Definition 2.1.11   A graph G = (V, E) is said to be k-connected if $1 \leq k < |V|$ and G has no separating set with fewer than k members.

Definition 2.1.12   A 2-connected graph is usually called a nonseparable graph.  (Thus, a nonseparable graph must have three or more vertices.)

Definition 2.1.13   A planar embedding of a graph G = (V, E) is a collection $G^* = (V^*, E^*)$ of points $(V^*)$ in the plane, and finite, non-self-intersecting curves $(E^*)$ in the plane, such that no two curves of $E^*$ meet except at endpoints, and such that

(1)   each node v of V can be associated with a point $v^*$ of $V^*$, and

(2)  each edge (v, w) of E can be associated with a curve of $E^*$ having endpoints $(v^*, w^*)$, so that

V, $V^*$ (respectively E, $E^*$) are thereby placed in 1-1 correspondence.

Definition 2.1.14  An abstract graph G is said to be planar if there exists a planar embedding of G.

Definition 2.1.15  Suppose $G^* = (V^*, E^*)$ is a planar embedding of G = (V, E). The faces of $G^*$ are the (necessarily open) connected components of the topological space $E^2 - (V^* \cup E^*)$ (where $E^2$ denotes the (Euclidean) plane). The boundary of a face is its topological boundary, when considered as a subspace of $E^2$. By Definition 2.1.13, this boundary corresponds to a subgraph of G, which (by abuse of language) is also called the boundary of the face. The periphery of $G^*$ is the boundary of the (unique) infinite face. (For greater amplification, see [Ore 67].)

Notation 2.1.16  The empty set is denoted by the symbol "$\phi$." The symbol "$\cup$" will be used for both set-theoretic union and graph-theoretic union; for example, if $G_i = (V_i, E_i)$, i = 1, 2, then "$G_1 \cup G_2$" denotes the graph with nodes $V_1 \cup V_2$ and edges $E_1 \cup E_2$.

## Section 2.  CONSTRUCTION OF (2, 1)-CONNECTED SEQUENCES

### 2.1  Definitions and Overview

The most important concept in this and succeeding chapters is that of (2, 1)-connected sequence.  The motivation behind this concept has already been discussed in Chapter 1.

Definition 2.2.1   A (2, 1)-connected sequence with respect to a non-separable graph G is a linear ordering $\langle v_1, v_2, \ldots, v_n \rangle$ of the set of nodes of G, such that, for all i, $1 < i < n$, either

(a)    the section graph of $\{v_1, \ldots, v_i\}$ is nonseparable (i.e., 2-connected), and

(b)    the section graph of $\{v_i, \ldots, v_n\}$ is connected (i.e., 1-connected)

or else $v_i$ is adjacent to exactly one node in $\{v_1, \ldots, v_{i-1}\}$ and to $v_{i+1}$.

As remarked in Chapter 1, the above definition embodies the essential properties underlying the somewhat unwieldy "circuit-star" decomposition idea.  The relationship between Definition 2.2.1 and "circuit-star" decomposition is most readily described with the aid of the following definition.

Definition 2.2.2   Suppose $S = \langle v_1, \ldots, v_n \rangle$ is a (2, 1)-connected sequence with respect to a nonseparable graph G.  Let $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ (where $i_j < i_{j+1}$) be the subsequence of S consisting of nodes v such that v is adjacent to two or more preceding nodes of S.  Let $S_1 = \langle v_1, \ldots, v_{i_1} \rangle$; $S_2 = \langle v_{i_1+1}, \ldots, v_{i_2} \rangle$; ...; $S_k = \langle v_{i_{k-1}+1}, \ldots, v_{i_k} \rangle$.  Each $S_i$ is called a segment of S, and the sequence $S_1, \ldots, S_k$ is called sequence of segments of S.

We might observe that if $S_1, \ldots, S_k$ is the sequence of segments of S, then $S_1 \circ \ldots \circ S_k = S$ (where " $\circ$ " denotes concatenation), since nonseparability of G implies $v_{i_k} = v_n$.

Associated with the sequence of segments is a decomposition of edges (G) which is very closely related to the "circuit-star" idea. Namely, let $G[S_i]$ denote the section graph of $S_i$ in G (Definition 2.1.9), and let

$$F_1 = G[S_1],$$

$$F_j = G[S_j] \cup \{\text{edges with one endpoint in } S_j \text{ and the other}$$
$$\text{in } S_1 \cup \ldots \cup S_{j-1}\} \quad (1 < j \le k)$$

It is proven in Chapter 3 (see Lemmas 3.3.8 and 3.3.7a) that: $F_1$ consists of an elementary circuit plus a set $E_1$ — where $E_1$ is a set of "extra" edges, all of which have endpoint $v_{i_1}$; further, $F_j$ ($j > 1$) consists of a "star" with center $v_{i_j}$ plus a set $E_j$ — where $E_j$ is a set of "extra" edges, all of which have one endpoint $v_{i_j}$ and the other endpoint lying in a single "ray" of the star. (For the formal definition of "star," see Definition 2.3.5.) To within the "extra" edges $E_1, \ldots, E_k$ (which cause no problem), the sequence $F_1, \ldots, F_k$ is a "circuit-star" decomposition: the condition (a) of Definition 2.2.1, coupled with the final clause of the definition, guarantees the circuit and star structure, and the condition (b) guarantees the "adjacency to a succeeding star" property mentioned in Chapter 1.

As explained in Chapter 1, the construction of (2, 1)-connected sequences is facilitated by the introduction of chain sequences.

2-7

<u>Definition 2.2.3</u>   Suppose $P = \langle v_1, \ldots, v_t \rangle$ is a path in G.  Let $G_p = (V_p, E_p)$ denote the subgraph of G composed of the nodes in P, and the edges in P (as defined in Definition 2.1.4).  A <u>chord</u> of P is an edge e in G, such that the endpoints of e belong to $V_p$, but e does not belong to $E_p$.  An <u>internal chord</u> is a chord which is not equal to $(v_1, v_t)$.

<u>Definition 2.2.4</u>   A <u>chain</u> is an elementary path with no internal chords.

Note that by Definition 2.1.5, an elementary path is not a circuit; therefore a chain is not a circuit.

<u>Definition 2.2.5</u>   A <u>chordless circuit</u> is an elementary circuit with no chords.

<u>Definition 2.2.6</u>   A <u>chain sequence</u> (CS) in a graph H is a sequence $r = \langle P, Q, \ldots, T \rangle$ of chains in H, where $P, \mathrm{int}(Q), \ldots, \mathrm{int}(T)$ are pair-wise disjoint, such that, for each S in r, other than P:

(1)     endpoints (S) lie in a preceding chain (more formally, endpoints $(S) \subseteq \cup \{\mathrm{nodes}(R) \mid R \underset{r}{\leq} S\}$, where "$R \underset{r}{\leq} S$" means R appears in r before S, reading from left to right) and

(2)     if $S \neq T$, there exists a node in int(S) adjacent to a node in the interior of a succeeding chain (more formally, $(\exists v \in \mathrm{int}(S))$ (v is adjacent to a node in $\cup \{\mathrm{int}(R) \mid R \underset{r}{>} S\})$).

A basic property of chain sequences, which follows directly from condition (2), is that every chain, other than the first, has non-empty interior.

Although a chain sequence is a sequence of paths, and a (2, 1)-connected sequence is a sequence of nodes, the two concepts are closely related.  Thus, condition (a) in Definition 2.2.1 is reflected in

requirement (1) of Definition 2.2.6, as well as in the requirement that the paths be chains; condition (b) of Definition 2.2.1 is reflected in condition (2) of Definition 2.2.6.

The incremental construction of a "complete" chain sequence (i.e., a chain sequence containing all the nodes of the graph in question) is simplified by the introduction of a slightly weaker concept, called a pre-chain sequence (pre-CS). While a pre-CS allows greater freedom during incremental construction, it has the valuable property that a "complete" pre-CS is necessarily a "complete" chain sequence, so we gain flexibility without suffering any loss in the result of the construction.

Definition 2.2.7   A pre-chain sequence (pre-CS) in a graph H is a sequence $r = \langle P, Q, \ldots, T \rangle$ of chains in H, where $P, int(Q), \ldots, int(T)$ are pair-wise disjoint, such that, for each S in r, other than P:

(1)     endpoints (S) lie in a preceding chain (i.e., endpoints
        $(S) \subseteq \bigcup \{ nodes(R) \mid R \underset{r}{<} S \}$) and

(2)     if $S \neq T$, there exists a node v in int(S) such that there is
        a path Z in H satisfying

        (2.1)   $int(Z) \subseteq nodes(H) - nodes(r)$, and

        (2.2)   one endpoint of Z is v, and the other belongs to
                the interior of a succeeding chain (i.e., to
                $\bigcup \{ int(R) \mid R \underset{r}{>} S \}$).

(We define "nodes (r)" to mean $\bigcup \{ nodes(R) \mid R \in r \}$.)

In the following informal discussion, we hope to provide some insight into the algorithm (T.1)-(T.5) (Section 2.2), whose objective is to generate a (2, 1)-connected sequence, given an arbitrary (planar or nonplanar) 3-connected graph G. The major portion of the algorithm

(steps (T.1)-(T.4.8)) consists of the construction of a "complete" pre-CS (ipso facto, a complete chain sequence) for G. The transition from a complete chain sequence to a (2, 1)-connected sequence (namely, step (T.5)) is comparatively simple (recall Chapter 1); therefore, we will concentrate on the "pre-CS" algorithm, (T.1)-(T.4.8).

The initialization step consists of generating a pre-CS $r = \langle R^1, R^2 \rangle$ as follows. First, a chordless circuit $C = \langle v_1, v_2, \dots, v_k, v_{k+1} \rangle$ (where $v_{k+1} = v_1$) is generated (for details, see the proof of Lemma 3.2.4 in Chapter 3). Then, set $R^1 = \langle v_1, v_2 \rangle$, and $R^2 = \langle v_2, \dots, v_{k+1} \rangle$. Designate the nodes in paths of r as "treated."

If all nodes of G are treated, then stop; otherwise, continue with (T.2)-(T.4.8). Steps (T.2)-(T.4.8) are devoted to "augmenting" $r$ — i.e., enlarging nodes(r) while preserving the pre-CS property — until all nodes of G are "treated." The augmentation is divided into cases. Let us begin by discussing the issues leading to this subdivision.

The augmentation process hinges on proving that a "judicious" choice of the path Q (illustrated in Figure 1.4, Chapter 1) can be made. The general proof technique is based, not on an analysis of each admissible path Q, but on an analysis of groups of such paths, called bridges. Informally, the bridge $B_v$ relative to an "untreated" node v (i.e., a node v not contained in nodes(r)) is the subgraph which is the union of all paths Q in G satisfying the following "admissibility" criteria:

(2.2.8)

        int(Q) $\subseteq$ untreated nodes, and

        endpoints (Q) $\subseteq$ treated nodes

and such that $v \in$ int(Q). (The idea of a bridge is well known (see [Ore 67], page 12ff.).) Bridges are often useful when dealing with the augmentation of a "treated" subgraph: the bridges tell how the "untreated"

portion of the graph "hangs together" — and it is precisely this kind of information in which we are interest, given our concern with connectivity.

The bridge concept is not used explicitly in the constructions, but it appears extensively in the proofs contained in Chapter 3. For the interested reader, we should point out that bridges appear in algorithm (T.1)-(T.4.8) under a simple, constructive guise as follows. Suppose $v$ is an untreated node. We need to make use of the set $\overline{A}(B_v)$ ("attachment nodes of $B_v$"), defined as nodes $(B_v) \cap$ (treated nodes), and the set $\text{int}(B_v)$ ("interior nodes of $B_v$"), defined as nodes $(B_v) \cap$ (untreated nodes). These two sets are simple to construct: generate the largest connected subgraph of G containing $v$, and all of whose nodes are un- treated; call it T. Then $\text{int}(B_v) = \text{nodes}(T)$, and $\overline{A}(B_v) = A(T)$, where $A(T)$ is defined as $\{w \in \text{nodes}(r) \,|\, w \text{ is adjacent to nodes}(T)\}$. (Note: references to proofs are deliberately omitted from the present informal discussion, in order that these comments may in no way be construed as being rigorous. Readers interested primarily in the formal development should proceed directly to the lemmas and theorems of Chapter 3.)

Now let us return to the description of the augmentation process. Suppose we are given a pre-chain sequence $r$, and suppose not all nodes of G are "treated." We wish to "augment" $r$, i.e., construct a pre-chain sequence $\bar{r}$ with nodes$(\bar{r}) \supsetneq$ nodes$(r)$. The general manner of augmen- tation has already been illustrated in Chapter 1, Figure 1.4: the idea is to select a path S in $r$, choose a bridge B with $\overline{A}(B) \cap \text{int}(S) \neq \phi$, make a suitable choice of path Q in B satisfying (2.2.8), with $\text{int}(Q) \neq \phi$, and finally apply one of the three constructions in Figure 1.4. The require- ments "$\overline{A}(B) \cap \text{int}(S) \neq \phi$" and "Q a path in B" are used later in proving that condition (2) of Definition 2.2.7 is satisfied; the requirement (2.2.8)

is used to verify condition (1) of the same definition; the requirement "int(Q) $\neq \phi$" is used to prove that nodes($\bar{r}$ ) $\supsetneq$ nodes(r).

Rather than choose an arbitrary S, we select the last path in r for which there exists a bridge B with $\overline{A}(B) \cap$ int(S) $\neq \phi$ (in computational terms: the last path S in r for which int(S) is adjacent to an untreated node). The primary reason for this choice is conceptual and computational simplicity: for such a path S, the condition (2) of Definition 2.2.7 is equivalent to the less complicated condition (2) of Definition 2.2.6, and this fact simplifies the task of proving that $S_1$ and $S_2$ (Figure 1.4) satisfy the conditions imposed by Definition 2.2.7. Let R denote this specially chosen path. Let B be any bridge with $\overline{A}(B) \cap$ int(R) $\neq \phi$.

The construction is divided into two cases, depending on whether $\overline{A}(B) \not\subseteq$ nodes(R) or $\overline{A}(B) \subseteq$ nodes(R). In keeping with the terminology of the construction (T.1)-(T.4.8), we will often refer to A(T) and nodes(T) instead of $\overline{A}(B)$ and int(B), where T is the subgraph discussed earlier. We will treat each case in turn.

Case 1.   A(T) $\not\subseteq$ nodes(R).

In this event, we certainly have A(T) $\not\subseteq$ int(R), so $\left| A(T)\text{-int}(R) \right| \geqslant 1$. There are two subcases.

Subcase 1.1.   $\left| A(T) - \text{int}(R) \right| \geqslant 2$.

Construction (1) of Figure 1.4 is carried out, where Q is a path in B satisfying the conditions discussed earlier. The endpoints of Q are selected to be any two distinct nodes in $\overline{A}(B) - \text{int}(R) = A(T) - \text{int}(R)$. The construction of Q takes advantage of a fundamental and useful property of the bridge B: namely, for any two distinct nodes v, w in

$\overline{A}(B)$, there is a path in B (of length greater than 1) from v to w, whose interior nodes belong to int(B); this property is essential to ensuring that nodes($\overline{r}$ )$\underset{\neq}{\supset}$ nodes(r).

Subcase 1.2.    $|A(T) - \text{int}(R)| = 1$.

Construction (2) of Figure 1.4 is carried out. However, considerable care must be taken to guarantee that the augmentation produces a pre-chain sequence. Two main problems arise: namely, ensuring that path $S_1$ (Figure 1.4(2)) is a chain, and ensuring that path $S_2$ satisfies condition (2) of Definition 2.2.7. Consider Figure 2.1.
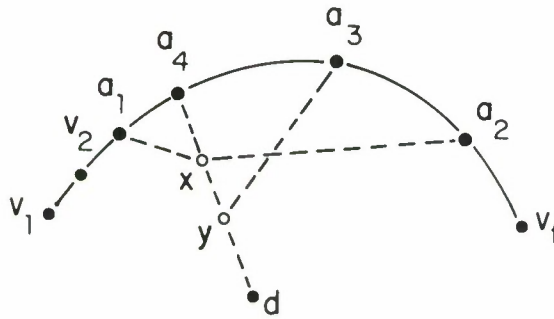


Figure 2.1   Example of a Bridge

For purposes of illustration, let the dotted edges denote the edges of B. Then $\overline{A}(B) = A(T) = \{a_1, a_2, a_3, a_4, d\}$; d is the unique member of $A(T) - \text{int}(R)$. In construction (2), we seek a path Q in B with d as one endpoint. Observe that the path $Q_1 = \langle a_4, x, y, d \rangle$, if utilized in construction (2), would produce an $S_1$ with internal chord $(a_1, x)$. To help avoid this problem, we stipulate that the other endpoint of Q (other than d) be chosen from the set $\{a_1, a_4\}$ (the first and last members of R

belonging to A(T) ). Suppose we choose $b=a_1$ as the other endpoint, and let $Q = \langle a_1, x, y, d \rangle$. This "extremal" choice of endpoint eliminates the kind of chord encountered with $Q_1$. However, chords can still arise in $S_1$, if, for example, d is adjacent to $v_2$.

The solution to this "d-adjacency" problem is illustrated in Figure 2.2, where $S_1$ is replaced by several paths.
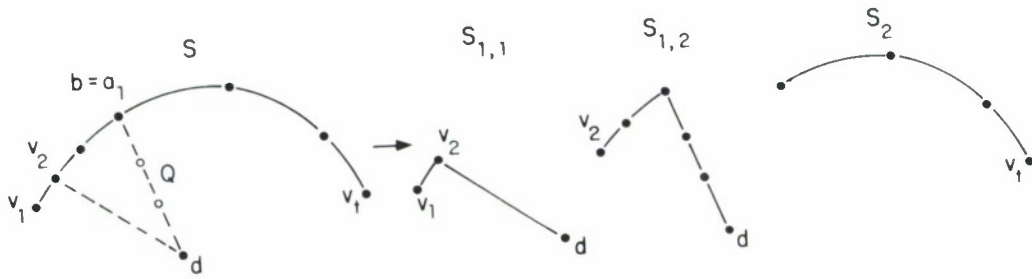


Figure 2.2    Solution to d-Adjacency

In general, $S_1$ is replaced by k+1 paths, where k is the number of nodes in $\text{int}\langle v_1, \ldots, a_1 \rangle \cup \{a_1\}$ adjacent to d. If d is adjacent to $a_1$, then the last of these k+1 paths will be Q itself. In this case, we have the further complication of ensuring that Q satisfies condition (2) of Definition 2.2.7. This particular problem is solved by a careful choice of endpoint b so as to guarantee that $\overline{A}(B) \cap \text{int}(S_2) \neq \phi$; this later inequality is shown to imply condition (2).

As noted above, the second main problem is to ensure that $S_2$ satisfies condition (2). Again, proper choice of endpoint b suffices to handle this problem.

We now come to the second main case.

Case 2.   $A(T) \subseteq nodes(R)$.

This case is divided into three subcases.  Let $a_1$ and $a_2$ denote the first and last nodes (respectively) of R which belong to A(T).  It will be shown (by 3-connectedness of G) that $|A(T)| \geq 3$,  so $a_1, a_2$ exist and are distinct; let $\langle a_1, \ldots, a_2 \rangle$ denote the subpath of R from $a_1$ to $a_2$. Let Q be any path in R with endpoints $a_1, a_2$,  and $\phi \neq int(Q) \subseteq int(B)$. (This "extremal" choice of endpoints is made for the same reason as in Subcase 1.2:  namely, avoidance of internal chords.)  The objective of Case 2 is either:  (i) to carry out construction (3) (Figure 1.4); or, if this construction fails to preserve the pre-chain sequence property, then the objective is:  (ii) to show how, through minor modification of r, or choice of a new bridge B,  it is possible to carry out construction (1), (2), or (3), while preserving the pre-chain sequence property.

If construction (3) is to preserve the pre-chain sequence property, then path $\langle a_1, \ldots, a_2 \rangle$ (which corresponds to the path $S_2$ in Figure 1.4(3)) must contain a node in its interior, adjacent to the interior of a succeeding chain (recall our earlier comment that, for path R, condition (2) of Definition 2.2.7 is equivalent to condition (2) of Definition 2.2.6). That is, we must have $int\langle a_1, \ldots, a_2 \rangle \cap s(R) \neq \phi$, where "s(R)" is defined below.

Definition 2.2.9   Given a pre-chain sequence r in a graph G, and a chain R in r, define s(R) (the "successor-adjacent" nodes of R) as follows:

$$s(R) \begin{cases} = \{v \in \text{int}(R) \mid (\exists S \underset{r}{>} R)(\exists w \in \text{int}(S)) \; v \text{ and } w \text{ are} \\ \qquad \text{adjacent in } G\} \quad \text{if } R \text{ is not the last chain in } r\,; \\ = \{\text{next-to-last node of } R\} \text{ if } R \text{ is the last chain in } r. \end{cases}$$

We can now state the first subcase.

Subcase 2.1   $\text{int}\langle a_1, \ldots, a_2 \rangle \cap s(R) \neq \phi$.

In this case, construction (3) (Figure 1.4) is carried out, with Q as described above.

Suppose, however, that the foregoing condition is not satisfied. It will be shown that, if $\text{int}\langle a_1, \ldots, a_2 \rangle$ contains a node adjacent to a node d in a preceding path, with $d \notin$ endpoints (R), then r can be modified so that Case 1 applies; we will discuss this modification below. In the following definition, we assign the notation "p(R)" to the set of interior nodes of R which are adjacent to some node d as just described.

Definition 2.2.10   Given a pre-chain sequence r in a graph G, and a chain R in r, define p(R) (the "predecessor-adjacent" nodes) of R as follows:

$$p(R) = \{v \in \text{int}(R) \mid (\exists S \underset{r}{<} R)(\exists w \in \text{int}(S)) \; v \text{ and } w \text{ are} \\ \qquad \text{adjacent, and } w \notin \text{endpoints (R)}\}.$$

Subcase 2.2   $\text{int}\langle a_1, \ldots, a_2 \rangle \cap s(R) = \phi$ and
$\text{int}\langle a_1, \ldots, a_2 \rangle \cap p(R) \neq \phi$.

Let b denote a node in the set $\text{int}\langle a_1, \ldots, a_2 \rangle \cap p(R)$. Consider Figure 2.3.
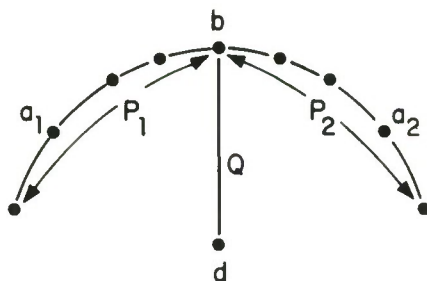
Figure 2.3    Illustration of Subcase 2.2

At least one of $P_1, P_2$ contains a node of $s(R)$ in its interior, since $s(R) \neq \phi$ (as will be proven), and $b \notin s(R)$. For the sake of discussion, let us suppose $\text{int}(P_2) \cap s(R) \neq \phi$. Then replace R by $R_1, R_2$ as indicated in Figure 2.4.
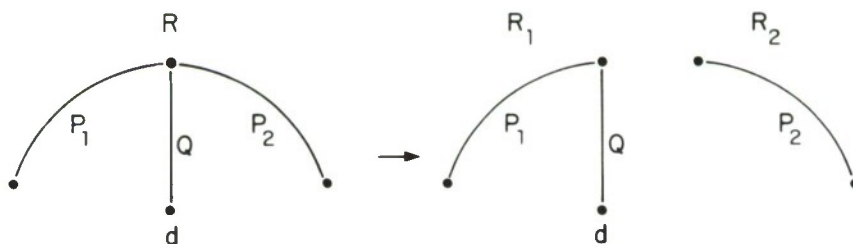


Figure 2.4    Replacement of R in Subcase 2.2

In addition, if d is adjacent to any nodes in $\text{int}(P_1)$ (which would produce internal chords in $R_1$), then replace $R_1$ by several paths, using the same method as suggested in the "solution to the d-adjacency problem" of Subcase 1.2 (see Figure 2.2).

After r has been modified in this fashion, the new "R" (i.e., the last path with an interior node adjacent to an untreated node) is path $P_2$.

Now, since $a_1, a_2 \in A(T)$ (see Figure 2.3), it follows that $A(T) \cap \text{int}(R) \neq \phi$ and $A(T) \not\subseteq \text{nodes }(R)$, so that Case 1 applies.

This completes Subcase 2.2.

<u>Subcase 2.3</u>    $\text{int}\langle a_1, \ldots, a_2 \rangle \cap s(R) = \phi$ and
$$\text{int}\langle a_1, \ldots, a_2 \rangle \cap p(R) = \phi.$$

In this case, a change of the bridge B is necessary; our task is to show that a bridge B' exists which will cause one of the preceding subcases to be applicable. To show the existence of such a bridge, we take advantage of the 3-connectedness of G. We consider the largest subpath $K = \langle c_1, \ldots, c_2 \rangle$ of R containing $\langle a_1, \ldots, a_2 \rangle$ and satisfying $\text{int}(K) \cap (s(R) \cup p(R)) = \phi$. The reason for introducing K is that its defining properties guarantee that

(2.2.11)     $\phi \neq \text{endpoints}(K) - \text{endpoints}(R) \subseteq s(R) \cup p(R)$ ;

the importance of this property will be discussed in a moment. It is proven that for such a K, the 3-connectedness of G implies that there exists a bridge B' satisfying

(2.2.12)
$$\begin{cases} \overline{A}(B') \cap \text{int}(K) \text{ and} \\ \overline{A}(B') \not\subseteq \text{nodes}(K) . \end{cases}$$

(Such a bridge can then be found by checking for "$\overline{A}(B_v) \not\subseteq \text{nodes}(K)$," relative to each bridge $B_v$, where v is an untreated node adjacent to $\text{int}(K)$.) Let T' be the subgraph corresponding to B', as discussed earlier; thus $A(T') = \overline{A}(B)$.

Now, if $A(T') \nsubseteq$ nodes(R), then Case 1 applies. On the other hand, if $A(T') \subseteq$ nodes(R), then (2.2.11) and (2.2.12) imply that $\text{int}\langle a_1', \ldots, a_2' \rangle$ meets $s(R) \cup p(R)$ (where $a_1'$ and $a_2'$ denote the first and last nodes (respectively) of R belonging to $A(T')$). (Intuitively: the nodes $a_1'$, $a_2'$ must surround at least one endpoint $c_i$ of K, where $c_i \notin$ endpoints (R) (see Figure 2.5) — otherwise, we would have $A(T') \subseteq$ nodes(K); by (2.2.11) p belongs to $s(R) \cup p(R)$.)
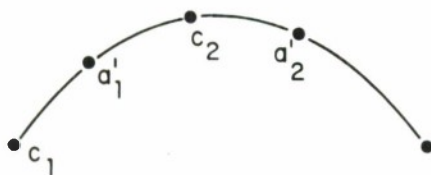


Figure 2.5   Illustration of Subcase 2.3

It follows that Subcase 2.1 or Subcase 2.2 is applicable (using subgraph T').

This completes the description of the algorithm for generating a chain sequence r, with nodes(r) = nodes(G). This algorithm is specified formally in steps (T.1)-(T.4.8), Section 2.2.

It may be helpful to the reader if we give a précis of the steps (T.1)-(T.4.8), taking advantage of the terminology used in the foregoing discussion.

Précis of algorithm (T.1)-(T.4.8) (Construction of "complete" chain sequence)

(T.1)   Initialization.

(T.2)   Test for completion;  start of augmentation.

Choice of path R and subgraph T.

(T.3)   Subdivision into Case 1 and Case 2 (for Case 2, go to T.4).

Case 1.   $A(T) \not\subseteq nodes(R)$.

Subdivision into Subcases 1.1 and 1.2.

(T.3.1)   Subcase 1.1   $|A(T) - int(R)| \geq 2$.

Construction (1), Figure 1.4 (Chapter 1).

(T.3.2)  Return to T.2.

(T.3.3)   Subcase 1.2   $|A(T) - int(R)| = 1$.

Construction (2), Figure 1.4.

(T.3.4)  Test for "d-adjacency" problem.

(T.3.5)  Treatment of "d-adjacency" problem.

(T.3.6)  Return to T.2.

(T.4)   Case 2.   $A(T) \subseteq nodes(R)$.

Definition of $a_1, a_2$.  Subdivision into Subcases 2.1, 2.2, and 2.3.

(T.4.1)   Subcase 2.1   $int \langle a_1, \ldots, a_2 \rangle \cap s(R) \neq \phi$.

Construction (3), Figure 1.4.

(T.4.2)  Return to T.2.

(T.4.3)   Subcase 2.2   $int \langle a_1, \ldots, a_2 \rangle \cap s(R) = \phi$  and

$$int \langle a_1, \ldots, a_2 \rangle \cap p(R) \neq \phi.$$

Modification of r.

(T.4.4)  Test for "d-adjacency" problem.

(T.4.5)  Treatment of "d-adjacency" problem.

(T.4.6)  Reduction to Case 1.

(T.4.7) <u>Subcase 2.3</u>  $\quad \text{int}\langle a_1, \ldots, a_2 \rangle \cap s(R) = \phi$ and

$$\text{int}\langle a_1, \ldots, a_2 \rangle \cap p(R) = \phi.$$

Choice of subgraph T'.

(T.4.8) Reduction to Case 1, Subcase 2.1, or

Subcase 2.2.

## 2.2 (2, 1)-Connected Sequences

The present subsection is devoted to a rigorous exposition of the algorithm for constructing a (2, 1)-connected sequence, given an arbitrary (planar or nonplanar) 3-connected graph G. The proof of the validity of this algorithm is to be found in Chapter 3, Section 2.2.

Algorithm (T.1)-(T.5)   (Construction of (2, 1)-connected sequence)

(T.1)   Choose an arbitrary chordless circuit (see Lemma 3.2.4) C in G. Suppose $C = \langle v_1, v_2, \ldots, v_k, v_{k+1} \rangle$ (where $v_{k+1} = v_k$). Let $R^1, R^2$ denote the paths $\langle v_1, v_2 \rangle$, $\langle v_1, \ldots, v_{k+1} \rangle$, respectively.
Let $r = \langle R^1, R^2 \rangle$.

(T.2)   Designate the nodes of r as <u>treated</u>. If all nodes of G are treated, go to T.5; otherwise, there exists at least one path S in r such that int(S) contains a node adjacent to an untreated node (as is proven later — see Theorem 3.2.20 and Lemma 3.2.11). Let R be the last such path in r. Construct any subgraph T that satisfies the following properties and is maximal with respect to them:

   (a)     T is connected;

   (b)     nodes(T) $\subseteq$ nodes(G) - nodes(r);

   (c)     nodes(T) is adjacent to int(R).

(It is proven later — see Theorem 3.2.20 and Lemma 3.2.12 — that such a T exists.)   Let $A(T) = \{w \in \text{nodes}(r) \,|\, w$ is adjacent to nodes(T)$\}$; thus $A(T) \cap \text{int}(R) \neq \phi$ by (c). It is proven later — see Theorem 3.2.20 and Lemma 3.2.13 — that $|A(T)| \geq 3$.

(T.3)   If $A(T) \subseteq \text{nodes}(R)$, then go to T.4; otherwise, proceed. Then $A(T) \not\subseteq \text{int}(R)$, so there are now two alternatives:   (1)   $A(T) - \text{int}(R)$

contains two or more nodes; (2) $A(T) - int(R)$ contains exactly one node (in which case this node does not belong to $nodes(R)$, since $A(T) \not\subseteq nodes(R)$). If alternative (1) holds, then go to T.3.1; if alternative (2) holds, then go to T.3.3.

(T.3.1)  By choice of case (see T.3 and T.4.6, alternative (1)), $A(T) - int(R)$ contains two or more nodes. Let $b, d$ be two such nodes. Choose a chain $Q$ from $b$ to $d$, with $\phi \neq int(Q) \subseteq nodes(T)$. (Existence of $Q$ will be proven later — see Theorem 3.2.20 and Lemma 3.2.15.)

Add $Q$ to $r$, immediately before path $R$.

(T.3.2)  Go to T.2.

(T.3.3)  By choice of case (see T.3 and T.4.6, alternative (2)), $A(T) - int(R)$ contains exactly one node. Let $d$ be the unique member of $A(T) - int(R)$. As noted in T.3, $d$ cannot belong to $R$. Let $a_1$ (respectively, $a_2$) be the first (respectively, last) node of path $R$ belonging to $A(T)$. Let $\langle a_1, \ldots, a_2 \rangle$ denote the subpath of $R$ from $a_1$ to $a_2$. (By the hypothesis of T.3.3 and the fact that $|A(T)| \geq 3$ (see T.2), it follows that $a_1, a_2$ exist and are distinct.)

Let $w$ be any node in $s(R)$. (It is proven later — see Theorem 3.2.20 and Lemma 3.2.16 — that $s(R) \neq \phi$.) Define $b$ as follows: if $w < a_2$ (relative to the order imposed by $R$), then let $b = a_2$; if $w \geq a_2$, then let $b = a_1$. Node $b$ divides $R$ into two edge-disjoint subpaths, and by the foregoing definition of $b$, the set $\{w\} \cup (\{a_1, a_2\} - \{b\})$ lies within the interior of (exactly) one of these; let $P_2$ denote this particular subpath, and let $P_1$ denote the other subpath. Choose a chain $Q$ from $b$ to $d$, with $\phi \neq int(Q) \subseteq nodes(T)$. (It is proven later — see Theorem 3.2.20 and

Lemma 3.2.15 — that such a chain exists.)  Let $P_1 \oplus Q$ denote the path formed by concatenating paths $P_1$ and $Q$ at the common node b.

Remove R from r and replace it with $P_1 \oplus Q$, followed immediately by $P_2$.

(T.3.4)   If $\text{int}(P_1) \cup \{b\}$ contains one or more nodes adjacent to d, then proceed; otherwise, go to T.2.

(T.3.5)   Let $f_1, \ldots, f_k$ be the nodes of $\text{int}(P_1) \cup \{b\}$ that are adjacent to d, in their order of appearance in $P_1 \oplus Q$ (thus, if $b \in \{f_1, \ldots, f_k\}$, then $b = f_k$).  Let $\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_{k+1}$ be the disjoint subpaths into which $P_1$ is broken by $f_1, \ldots, f_k$ (so that (last node in $\overline{P}_i$) = (first node in $\overline{P}_{i+1}$) = $f_i$, for i = 1, ..., k).  (If $f_k = b$, then $\overline{P}_{k+1}$ is empty.)  Let $e_i$ denote the edge $(f_i, d)$, (i = 1, ..., k).

Replace $P_1 \oplus Q$ by the following sequence of k+1 paths:
$$\overline{P}_1 \oplus e_1, \ \overline{P}_2 \oplus e_2, \ \ldots, \ \overline{P}_k \oplus e_k, \ \overline{P}_{k+1} \oplus Q.$$

(T.3.6)   Go to T.2.

(T.4)     By choice of case, $A(T) \subseteq \text{nodes}(R)$.  Let $a_1$ (respectively, $a_2$) be the first (respectively, last) node of path R belonging to A(T).   Let $\langle a_1, \ldots, a_2 \rangle$ denote the subpath of R from $a_1$ to $a_2$.  (Because $A(T) \subseteq$ nodes(R) and $|A(T)| \geq 3$ (see T.2), it follows that $a_1, a_2$ exist and are distinct, and $\text{int}\langle a_1, \ldots, a_2 \rangle \neq \phi$.)  There are now three alternatives:

(1)   $\text{int}\langle a_1, \ldots, a_2 \rangle$ meets s(R);

(2)   $\text{int}\langle a_1, \ldots, a_2 \rangle$ does not meet s(R), but does meet p(R);

(3)   $\text{int}\langle a_1, \ldots, a_2 \rangle$ meets neither s(R) nor p(R).

In alternative (1), go to T.4.1; in alternative (2), go to T.4.3; in alternative (3), go to T.4.7.

(T.4.1) By choice of case (see T.4 or T.4.8, alternative (1)), $A(T) \subseteq$ nodes(R), and $\text{int}\langle a_1, \ldots, a_2 \rangle$ meets s(R) (this fact will be needed in later proofs). Let $R_1, R_2, R_3$ denote the sequence of edge-disjoint subpaths into which R is divided by $a_1, a_2$. (Thus, $R_2 = \langle a_1, \ldots, a_2 \rangle$; path $R_1$ (or $R_3$) will be empty if $a_1$ (or $a_2$) is an endpoint of R.) Choose a chain Q from $a_1$ to $a_2$, with $\phi \neq \text{int}(Q) \subseteq \text{nodes}(T)$. (It is proven later — see Theorem 3.2.20 and Lemma 3.2.15 — that such a chain exists.) Let $R_1 \oplus Q \oplus R_3$ denote the path formed by concatenating $R_1$, Q, and $R_3$.

Replace R by $R_1 \oplus Q \oplus R_3$, followed by path $R_2$.

(T.4.2) Go to T.2.

(T.4.3) By choice of case (see T.4 or T.4.8, alternative (2)), $p(R) \cap \text{int}\langle a_1, \ldots, a_2 \rangle$ is nonempty. Let b be any node in this intersection. Choose any node d adjacent to b and belonging to some $P \underset{r}{<} R$, with $d \notin \text{endpoints}(R)$. (Such a node exists, by definition of p(R).) Let $Q = \langle b, d \rangle$. Let w be any node in s(R). (It will be proven later — see Theorem 3.2.20 and Lemma 3.2.16 — that s(R) is nonempty.) By hypothesis (see T.4 or T.4.8, alternative (2)), $w \neq b$. Hence, b divides R into two edge-disjoint subpaths, exactly one of which contains w in its interior. Let $P_1$ (respectively, $P_2$) = whichever of these subpaths does not (respectively, does) contain w in its interior. Let $P_1 \oplus Q$ denote the path formed by concatenating paths $P_1$ and Q at the common node b, taken in the direction such that the last node is d.

Remove R from r and replace it with $P_1 \oplus Q$, followed immediately by $P_2$. Let R now denote $P_2$. (T and $R = P_2$ satisfy the conditions in step T.2, as is easily checked.) Note that A(T) meets $\text{int}(P_1)$, so that $A(T) \not\subseteq \text{nodes}(P_2) = \text{nodes}(R)$.

(T.4.4)   If $\text{int}(P_1)$ contains one or more nodes adjacent to d, then proceed; otherwise, go to T.4.6.

(T.4.5)   Let $f_1, \ldots, f_k$ be the nodes of $\text{int}(P_1)$ that are adjacent to d, in their order of appearance in $P_1 \oplus Q$. Let $\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_{k+1}$ be the disjoint subpaths into which $P_1$ is broken by $f_1, \ldots, f_k$ (so that (last node in $\overline{P}_i$) = (first node in $\overline{P}_{i+1}$) = $f_i$, for $i = 1, \ldots, k$). Let $e_i$ denote the edge $(f_i, d)$, $(i = 1, \ldots, k)$.

   Replace $P_1 \uplus Q$ by the following sequence of k+1 paths:
$\overline{P}_1 \uplus e_1, \ldots, \overline{P}_k \oplus e_k, \ \overline{P}_{k+1} \oplus Q.$
   Continue.

(T.4.6)   Now $A(T) \subseteq \text{nodes}(R)$ (as noted at the end of T.4.3), so there are two alternatives:

   (1)   $A(T)-\text{int}(R)$ contains two or more nodes;

   (2)   $A(T)-\text{int}(R)$ contains exactly one node (in which case this node is not a node of R, since $A(T) \subseteq \text{nodes}(R)$).

In alternative (1), go to T.3.1; in alternative (2), go to T.3.3. (As an aside, it might be mentioned that in this case, T.3.5 will not be applicable — otherwise, R would not be a chain, and it will be shown in Theorem 3.2.20 that every path in r is a chain.)

(T.4.7)   By choice of case (see T.4, alternative (3)) $A(T) \subseteq \text{nodes}(R)$, and $\text{int}\langle a_1, \ldots, a_2 \rangle$ meets neither $s(R)$ nor $p(R)$. Let K = longest subpath of R which contains $\langle a_1, \ldots, a_2 \rangle$ and whose interior meets neither $s(R)$ nor $p(R)$. As will be proven later — see Theorem 3.2.20 and Lemma 3.2.18 — the 3-connectedness of G implies that there exists a subgraph T′ which satisfies, and is maximal with respect to, properties (a), (b), (c) (see T.2), and such that: (d) $A(T')$ meets $\text{int}(K)$ and (e) $A(T') \not\subseteq \text{nodes}(K)$.

Let T be any such subgraph. In particular, then, R and this T satisfy the conditions in T.2.

(T.4.8)  If $A(T) \not\subseteq nodes(R)$, then go to T.3.  If $A(T) \subseteq nodes(R)$, then let $a_1$ (respectively, $a_2$) = first (respectively, last) node, of path R belonging to $A(T)$; let $\langle a_1, \ldots, a_2 \rangle$ denote the subpath of R from $a_1$ to $a_2$.  (Since $A(T) \subseteq nodes(R)$ and $|A(T)| \geq 3$ (see T.2), it follows that $a_1$, $a_2$ exist and are distinct.)  Then $int\langle a_1, \ldots, a_2 \rangle$ meets $s(R) \cup p(R)$, by properties (d), (e), and maximality of K.  Thus there are two alternatives:

    (1)   $int\langle a_1, \ldots, a_2 \rangle$ meets $s(R)$;

    (2)   $int\langle a_1, \ldots, a_2 \rangle$ does not meet $s(R)$, but does meet $p(R)$.

In alternative (1), go to T.4.1; in alternative (2), go to T.4.3.

(T.5)  By choice of case (see T.2), all the nodes of G are now "treated," i.e., belong to paths in r.  If the first node of the first path is identical to the first node of the second path, then "reverse" the second path (i.e., replace it by the path composed of its nodes, taken in reverse order).  Let $r = \langle F^0, F^1, \ldots, F^q \rangle$ denote the resulting sequence.  Let $\langle v_i^1, \ldots, v_i^{t(i)} \rangle$ denote the sequence of nodes comprising $F^i$ ($i \geq 0$).

    (It will be shown in Theorem 3.2.20 that r is a chain sequence; hence, by condition (2) of Definition 2.2.6, $s(F^i) \neq \phi$, for $i = 1, \ldots, q$.)  Let $v_i^{s(i)}$ denote the last node of $\langle v_i^1, \ldots, v_j^{t(i)} \rangle$ belonging to $s(F^i)$ ($i = 1, \ldots, q$).  For $i = 1, \ldots, q$, define $F_{(1)}^i = \langle v_i^2, v_i^3, \ldots, v_i^{s(i)} \rangle$ and $F_{(2)}^i = \langle v_i^{t(i)-1}, v_i^{t(i)-2}, \ldots, v_i^{s(i)+1} \rangle$ (if $s(i) = t(i)-1$, then $F_{(2)}^i$ is empty).

    Let S denote the sequence of nodes $F^0 \circ F_{(2)}^1 \circ F_{(1)}^1 \circ \ldots \circ F_{(2)}^q \circ F_{(1)}^q$ (where "$\circ$" denotes concatenation of nodes).  (It is proven in Chapter 3, Section 2.2 (see Theorems 3.2.20 and 3.2.21) that S is a (2, 1)-connected sequence with respect to G.)

## 2.3  Primitive (2, 1)-Connected Sequences

This subsection introduces a special class of (2, 1)-connected sequences, the "primitive"(2, 1)-connected sequences. These sequences have two practical advantages over arbitrary (2, 1)-connected sequences. First, they permit the application of a much simpler planar embedding algorithm (i.e., planarity test) than that associated with arbitrary sequences. Second, they permit the construction of a planar embedding (for a planar graph) containing an arbitrary pre-selected edge on the periphery. The second property is particularly valuable whenever two planar embeddings sharing a single common edge are to be "pasted together" — an operation that is fundamental to any application of the present methods to nonseparable graphs (see [MacL37b]).

The underlying idea of a primitive (2, 1)-connected sequence is the concept of a "primitive" circuit, which we define as a chordless circuit whose nodes do not constitute a separating set. The essential property of such circuits is that for a 3-connected graph, such a circuit must be a face in any planar embedding of the graph (see [MacL37b], page 466, Theorem 6); this fact is later used to advantage in Lemma 3.3.9.

Definition 2.2.13   A primitive (2, 1)-connected sequence with respect to a nonseparable graph G is a (2, 1)-connected sequence $\langle v_1, \ldots, v_n \rangle$ such that G contains a "primitive" circuit through $v_n$ and the edge $(v_1, v_2)$ — where by a "primitive" circuit we mean a chordless circuit whose nodes are not a separating set in G (see Definition 2.1.10).

The following algorithm describes the construction of a primitive (2, 1)-connected sequence, given an arbitrary (planar or nonplanar) 3-connected graph G. The proof of the validity of this algorithm is given

in Chapter 3, Section 2.3.

Algorithm (P.1)-(P.3)  (Construction of primitive (2, 1)-connected sequence)

(P.1)   Apply algorithm (T.1)-(T.5) to G.  Let $S = \langle v_1, \ldots, v_n \rangle$ denote the resulting sequence.

(P.2)   Let $v_{i_0}$ be the first node v in S  such that v is adjacent to two or more preceding nodes in S.  (Such a node exists, since S  contains all nodes of G, as will be proven in Theorem 3.2.20, and G is 3-connected.) Let $C = \langle v_1, \ldots, v_{i_0}, v_1 \rangle$.  (It will be proven in Theorem 3.2.23 that C is a chordless circuit.)

(P.3)   Repeat algorithm (T.1)-(T.5), utilizing the above circuit C in step (T.1).  (It should be emphasized that in steps (T.3.3) and (T.4.3), the nodes in the subpath $P_2$ are to appear in the same order as they do in R — as implied by the definition of "subpath" (Definition 2.1.4).  The same remark applies to subpath $R_2$ in step (T.4.1).)  Let $\overline{S}$ denote the resulting sequence.  (It will be proven in Theorem 3.2.23 that $\overline{S}$ is a primitive (2, 1)-connected sequence with respect to G.)

Section 3. EXPANSION ALGORITHM

3.1 Definitions and Overview

The purpose of the present section is to describe how a $(2, 1)$-connected sequence with respect to a graph G can be used simultaneously to test G for planarity and to construct a planar embedding for G (when G is planar). The algorithm developed for this purpose is called the Expansion Algorithm; it is a formalization of the construction method illustrated in Figure 1.2, Chapter 1. The salient feature of the Expansion Algorithm is that it generates a planar embedding "from the interior out"; that is, starting with a $(2, 1)$-connected sequence $\langle v_1, \ldots, v_n \rangle$, the algorithm constructs a planar embedding by placing successive nodes of the sequence outside the periphery of the previously constructed portion of the graph (see Figure 1.2). It is this feature that gives the algorithm its name.

The task of this subsection is to introduce sufficient terminology to enable us to describe the Expansion Algorithm in a precise manner; in addition, we will attempt to motivate the details of the algorithm. The formal description will be given in Sections 3.2 and 3.3. We will begin, for the sake of simplicity, with the version of the Expansion Algorithm which applies to primitive $(2, 1)$-connected sequences (see Section 2.3); this version is called the Expansion Algorithm/Linear Version.

Informally, the Expansion Algorithm/Linear Version (abbreviated EA/L) proceeds as follows. The initial step is to construct a planar embedding of the graph $F_1 = [S_1]$ (see Section 2.1), as in Figure 2.6.
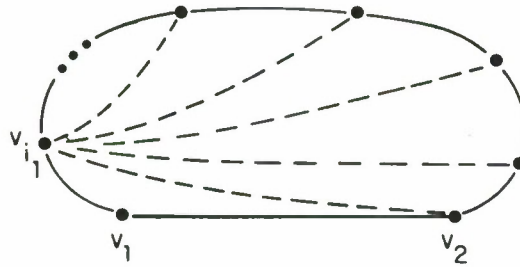
Figure 2.6  Initial Step of Expansion Algorithm/Linear Version

As explained in Section 2.1, $F_1$ consists of an elementary circuit plus a set of "extra" edges emanating from a common node; thus, informally speaking, $F_1$ always "looks like" the graph illustrated in Figure 2.6; in particular, $F_1$ is planar. Let $G_1^*$ denote the planar embedding of $F_1$. The algorithm now proceeds by iteration of the following step. Suppose $G_j^*$ has been constructed ($j \geq 1$). If $j = k$, then STOP: the graph G is planar, and $G_j^* = G_k^*$ is a planar embedding of G; if $j < k$, then we check whether the "expansion" step illustrated below in Figure 2.7 can be carried out. (As discussed in Section 2.1, $F_j$ ($j > 1$) consists of a "star" plus a set of "extra" edges which emanate from the center of the star, and whose other endpoints all lie on a single "ray" of the star; these extra edges are suggested by dotted lines in Figure 2.7.)  This check simply consists of observing whether all the "endpoints" of $F_{j+1}$ lie on the periphery of $G_j^*$ (where by the "endpoints" of $F_{j+1}$, we mean the nodes of $F_{j+1}$ belonging to $S_1 \cup \ldots \cup S_j$ — that is, to the set of nodes of $G_j^*$). If this check fails, then G is nonplanar; if the check succeeds, then we carry out the "expansion" illustrated in Figure 2.7 and call the result $G_{j+1}^*$. Notice that the edge $(v_1, v_2)$ lies on the periphery of $G_{j+1}^*$.
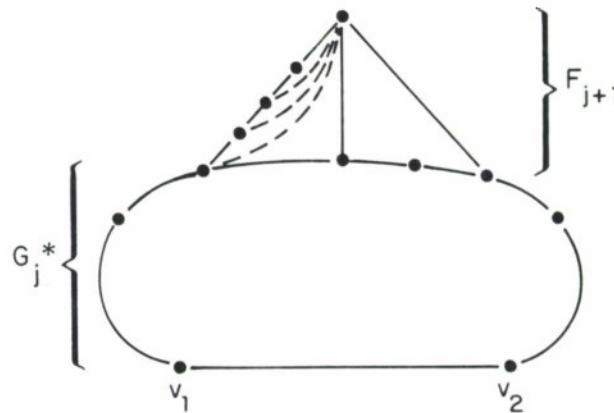
Figure 2.7   Typical Intermediate Step of Expansion
Algorithm/Linear Version

In order to formalize and prove the validity of the algorithm just
described, we need to introduce some precise terminology.  Foremost
among the items needing formalization is the "expansion" step illus-
trated in Figure 2.7.  An example will help to motivate the details of
this formalization.  Essentially, we are concerned with the idea of ex-
panding a periphery (namely, an elementary circuit), given a "star"
whose "endpoints" lie on this periphery.  An essential point is that
there is more than one way to go about such an expansion.  Consider the
example in Figure 2.8, where we have a star S with three endpoints.  As
illustrated, there are three essentially different ways of expanding the
circuit C.  Indeed, if we consider two planar embeddings to be identical
if they have the same face boundaries and the same periphery (when
regarded as abstract circuits) then there are exactly three different
ways of "expanding" the circuit C to a planar embedding of $C \cup S$ (i.e.,
to a planar embedding with node g on the periphery).  If the above star
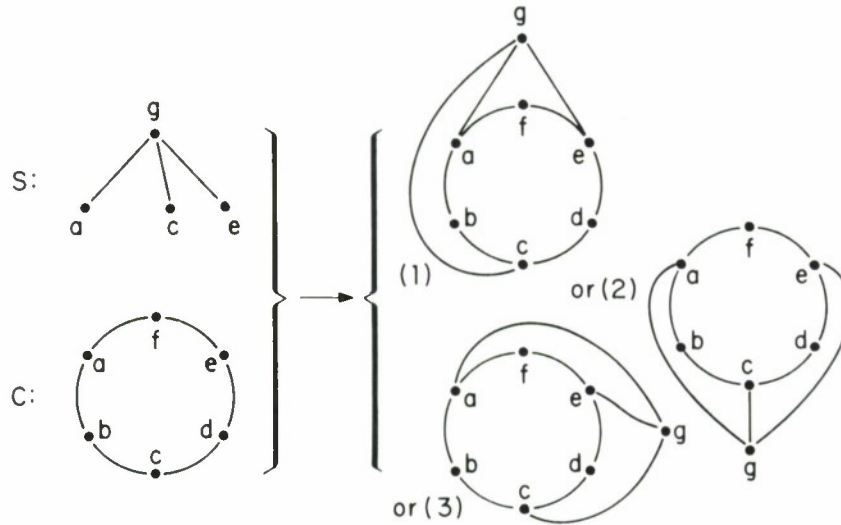
Figure 2.8   Expansion of a Circuit Relative to a Star

had k endpoints, then there would be k different expansions.  In order
to specify which expansion we desire to construct, we introduce the
idea of a "base."  Thus, we say that the expansion in Figure 2.8 (1) is
"based on edge (c, d)" (or on edge (d, e) ); that is, by specifying an edge
of C which lies on the periphery of the expansion, we uniquely specify
one of the k possible expansions.

The above terminology enables us to describe also the version of
the Expansion Algorithm which is applicable to arbitrary (2, 1)-
connected sequences; this version will be called the "Expansion
Algorithm/Peripheral Version" (abbreviated EA/P).  The initial step of
EA/P is the same as the initial step of EA/L (see Figure 2.6).  The
typical intermediate step is as follows.  Suppose $G_j^*$ has been constructed
(j ≥ 1).  If j = k, then STOP:  the graph G is planar, and $G_j^* = G_k^*$ is a

planar embedding of G; if $j < k$, then we check whether an "admissible" expansion can be carried out. The check for an "admissible" expansion is the following (first, we must observe that unlike EA/L, the algorithm EA/P guarantees that the "endpoints" of $F_{j+1}$ are a subset of the set of nodes on the periphery of $G_j^*$): is there an expansion based on some edge in periphery $(G_j^*)$, such that the periphery of this expansion contains all "deficient nodes of $F_1 \cup \ldots \cup F_{j+1}$" (where such a node is defined as a node adjacent to nodes $(F_{j+2} \cup \ldots \cup F_k))$? If so, then we carry out such an expansion. (The foregoing check can be done conveniently by examining each of the subpaths into which periphery $(G_j^*)$ is divided by the endpoints of $F_{j+1}$; however, we will not enter into such details here.)

In summary, the main difference between EA/L and EA/P is that in EA/P, we must repeatedly search for an acceptable "base" for the expansion step, whereas in EA/L, the same base (namely, edge $(v_1, v_2)$) is used throughout (see Figure 2.7). Thus the algorithm EA/L produces a planar embedding with $(v_1, v_2)$ on the periphery (whenever G is planar). The advantages of this fact have already been mentioned in Section 2.3.

Let us briefly précis the steps of the Expansion Algorithm, as they appear in Sections 3.2 and 3.3.

Précis of Algorithm (L.1)-(L.2.2)    (Expansion Algorithm/Linear Version)

(L.1) Initialization:  generate $G_1^*$; set $e_0$ equal to edge $(v_1, v_2)$.

(L.2) Iteration step:

(i)    Check whether "endpoints" lie on periphery $(G_j^*)$ (if not, STOP);

(ii)  If there are no "extra" edges (see above discussion), go to (L.2.1); otherwise go to (L.2.2).

(This breakdown is made for purely expository reasons: i.e., so that we can give the essential picture in (L.2.1), then advance to the simple but verbally cumbersome treatment of the "extra" edges in (L.2.2).  The need for this breakdown stems from our formal definition of "expansion" purely in terms of stars;  see Definition 2.3.6.)

(L.2.1)  Generate an "expansion," based on edge $e_0$ (without need to treat "extra" edges).

(L.2.2)  Generate an "expansion," based on edge $e_0$ (taking "extra" edges into account).


## Précis of Expansion Algorithm/Peripheral Version

This algorithm is the same as the foregoing, except that (L.2) (i) is replaced by the following:

"(i)  Check whether there exists an "admissible" expansion (if not, STOP).  Set $e_0$ equal to the "base" of this expansion."

With the above motivation in mind, let us proceed to some formal definitions.  Recall the definitions of (2, 1)-connected sequence and segment (Definitions 2.2.1 and 2.2.2).

The next three definitions will help us to discuss the connection between (2, 1)-connected sequences and "stars" (see Section 2.1).

Definition 2.3.1  A segment of a (2, 1)-connected sequence is said to be singular if it contains exactly one node; otherwise it is said to be

2-35

nonsingular.

Definition 2.3.2   Suppose $S_i = \langle w_1, \ldots, w_t \rangle$, $i > 1$ is a nonsingular segment of a $(2, 1)$-connected sequence S.  Then by definition, $w_1$ must be adjacent to exactly one preceding node in S (see remark following this definition).  This unique node is called the anchor of $S_i$ (denoted "anchor $(S_i)$").

Remark on Definition 2.3.2   Since $S_i$ is a nonsingular segment, $w_1$ cannot be adjacent to more than one preceding node, and by definition of $(2, 1)$-connectedness, $w_1$ must be adjacent to at least one preceding node.

Definition 2.3.3   Suppose $S_i = \langle w_1, \ldots, w_t \rangle$ is a segment of a $(2, 1)$-connected sequence S.  By Base $(S_i)$ we mean the set of all nodes v preceding $w_1$ in S, such that v is adjacent to a member of $S_i$.  (Thus Base $(S_i) = \phi$.)

The next definition is a formalization of the notion of "expansion" discussed informally above.

Definition 2.3.4   Suppose C is an elementary circuit in a graph G, $C^*$ is a planar embedding of C, e is an edge of C, and $e^*$ is the corresponding line segment of $C^*$.  Suppose v belongs to nodes(G)-nodes(C), and let $E_C(v)$ denote the subgraph composed of the edges in the set {edges of G with one endpoint v and the other endpoint in C}.  Suppose $E_C(v)$ contains two or more nodes of C.

By a linear expansion of $C^*$ (based on e) relative to v, we mean any extension of $C^*$ to a planar embedding of $C \cup E_C(v)$, such that

1)    $v^*$ lies outside $C^*$ (where $v^*$ denotes the geometric repre-
      sentation of v) and

2)    $e^*$ lies on the periphery of this embedding.

Remark on Definition 2.3.4    It should be observed that such an exten-
sion of $C^*$ exists, by fundamental topological properties of the plane.

For example, suppose G is a graph containing a circuit
$C = \langle a, b, c, d, f, g, a \rangle$ and a node $v \notin$ nodes(C); suppose $E_C(c) = \{(v,a), (v,g),$
$(v,f), (v,d)\}$.  Let e = edge (c, d).  If $C^*$ is the embedding shown in
Figure 2.9(a), then Figure 2.9(b) represents a linear expansion of $C^*$
(based on e = (c, d) relative to v.
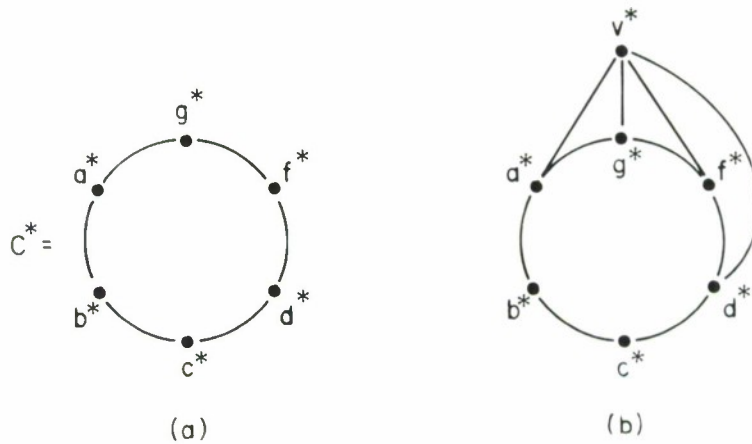


Figure 2.9    Linear Expansion

Observe that the periphery of Figure 2.9(b) may be produced from $C^*$ by
holding $e^* = (c^*, d^*)$ fixed and expanding outward some suitable portion of
the "line" $C^* - e^*$.  The term "linear expansion" is based on this aspect
of the construction.

For subsequent applications, the idea of linear expansion must be
extended to include not only single points v, but also suitable paths

$W = \langle w_1, \ldots, w_t \rangle$. Informally, a "suitable" path is one for which the set $E_C(W) = \{$edges with one node in W and one node in C$\} \cup \{$edges with both nodes in W$\}$ "looks essentially like" the set $E_C(v)$ in Definition 2.3.4. These informal ideas are made more precise in the following definitions.

Definition 2.3.5   A <u>simple</u> <u>star</u> with <u>center</u> v is a graph G = (V, E) such that E is of the form $\{(v, v_i) \mid i=1, \ldots, k\}$ (where $v_i \neq v$). A <u>star</u> with center v is a graph that is either a simple star with center v, or else can be generated from a simple star with center v by "addition of nodes of degree 2" (i.e., the successive replacement of an arbitrary edge $(v_i, v_j)$ by two edges $(v_i, w)$, $(w, v_j)$, where w is a new node).

Definition 2.3.6   Suppose C is an elementary circuit in a graph G, $C^*$ is a planar embedding of C, e is an edge of C, and $e^*$ is the corresponding line segment of $C^*$. Suppose $W = \langle w_1, \ldots, w_t \rangle$ is a path in G, whose nodes belong to nodes(G)-nodes(C), and let $E_C(W) = \{$edges of G with one node in W and one node in C$\} \cup \{$edges with both nodes in W$\}$. Suppose that $E_C(W)$ contains two or more nodes of C and that $E_C(W)$ is a star with center $w_t$.

By a <u>linear</u> <u>expansion</u> of $C^*$ (based on e) relative to W, we mean any extension of $C^*$ to a planar embedding of $C \cup E_C(W)$, such that

1)   $W^*$ lies outside $C^*$ (where $W^*$ denotes the set of points in the plane, corresponding to the nodes in W), and

2)   $e^*$ lies on the periphery of this embedding.

Remark on Definition 2.3.6   It should be observed that such an extension always exists, under the hypothesis on $E_C(W)$.

An example of this general type of "linear expansion" would look very much like Figure 2.9, except that one of the edges added in Figure 2.9(b) would be replaced by an elementary path (node-disjoint from the other added edges — except for the "center" node).

## 3.2 Expansion Algorithm/Linear Version

· This subsection contains the formal description of the Expansion Algorithm/Linear Version. For its motivation, see Section 3.1. The formal proof of its validity appears in Chapter 3, Section 3.2. Throughout the present section, we assume that G is a 3-connected graph and that S is a primitive (2, 1)-connected sequence with respect to G. Let $S_1, \ldots, S_k$ be the sequence of segments of S (see Definition 2.2.2).

Algorithm (L.1)-(L.2.2)    (Expansion Algorithm/Linear Version)

(L.1)    Let $C_1 = \langle v_1, \ldots, v_t, v_1 \rangle$ where $\langle v_1, \ldots, v_t \rangle = S_1$. (It will be proven in Theorem 3.2.23 that $C_1$ is an elementary circuit.) Let $E_1 = \{e \in \text{edges}(G) \mid e = (v_t, v_i) \text{ and } i \in \{2, \ldots, t\text{-}2\}\}$. (Thus $C_1 \cup E_1$ is planar.) Let $G_1^* = $ any planar embedding of $C_1 \cup E_1$ with periphery $C_1$. Let $C_1^*$ denote the restriction of $G_1^*$ to $C_1$. Let $e_0 = \text{edge}(v_1, v_2)$. Let $j = 1$.

(L.2)    If $j = k$, STOP (G is planar and $G^* = G_k^*$ is a planar embedding of G, as will be proven in Theorem 3.3.12); otherwise proceed. If $\text{Base}(S_{j+1}) \not\subseteq \text{nodes}(C_j)$ (see Definition 2.3.3) then STOP (G is nonplanar); otherwise proceed. Let $j = j + 1$.

Let $\langle v_1, \ldots, v_t \rangle$ denote the sequence of vertices of $S_j$. If $S_j$ is non-singular and $v_t$ is adjacent to a node in $\{\text{anchor}(S_j), v_1, \ldots, v_{t-2}\}$ (see Definition 2.3.2), then go to L.2.2; otherwise, go to L.2.1.

(L.2.1)  By hypothesis, either $S_j$ is singular, or $S_j$ is nonsingular and $v_t$ is not adjacent to any node in $\{\text{anchor}(S_j), v_1, \ldots, v_{t-2}\}$.

Let $G_j^* =$ any planar embedding obtained from $G_{j-1}^*$ by replacing $C_{j-1}^*$ with a linear expansion of $C_{j-1}^*$ (based on $e_0$) relative to $S_j$ (see Definition 2.3.6).  (It will be proven in Theorem 3.3.12 that under the hypothesis of L.2.1, the conditions of Definition 2.3.6 are satisfied and a "linear expansion" exists.)

Let $C_j^* =$ periphery of $G_j^*$, and let $C_j$ denote the underlying graph of $C_j^*$.  Go to L.2.

(L.2.2)  By hypothesis, $S_j$ is nonsingular, and there exist edges in G with one endpoint $= v_t$ and the other endpoint in $\{\text{anchor}(S_j), v_1, \ldots, v_{j-2}\}$. Let $\overline{E}$ denote the set of all such edges.

Temporarily replace G by $G-\overline{E}$ and let $G_j^* =$ any planar embedding obtained from $G_{j-1}^*$ by replacing $C_{j-1}^*$ with a linear expansion of $C_{j-1}^*$ (based on $e_0$) relative to $S_j$.  (With the removal of $\overline{E}$, the conditions of Definition 2.3.6 are satisfied.)  Let $C_j^* =$ periphery of $G_j^*$, and let $C_j$ denote the underlying graph of $C_j^*$.

Now, restore edges $\overline{E}$ to G.  Add corresponding line segments $\overline{E}^*$ to $G_j^*$ in planar fashion, inside $C_j^*$.  (Such an addition is always possible, as will be proven in Theorem 3.3.12.)  Thus the periphery of $G_j^*$ remains $C_j^*$.

Go to L.2.

## 3.3  Expansion Algorithm/Peripheral Version

This subsection contains the formal description of the Expansion Algorithm/Peripheral Version;  see Section 3.1 for a discussion of its motivation.  The proof of the validity of this algorithm is given in Chapter 3, Section 3.3.  Throughout the present section, it is assumed that G is a nonseparable graph and S is a (2, 1)-connected sequence with respect to G.  (It is not assumed that G is 3-connected, inasmuch as 3-connectedness is not required in the formal proof in Chapter 3, Section 3.3.)

The following definitions will help to simplify the presentation.

Definition 2.3.7   Suppose S is a (2, 1)-connected sequence for G,  with segments $S_1, S_2, \ldots, S_k$.  Suppose $j > 1$, $S_j = \langle w_1, \ldots, w_t \rangle$ and $x \in \text{Base}(S_j)$. Define

$$
_xS_j = \begin{cases} \langle x, w_t \rangle & \text{if } t = 1 \\ \langle x, w_1, \ldots, w_t \rangle & \text{if } t > 1 \text{ and } x = \text{anchor}(S_j) \text{ (see Definition 2.3.2)} \\ \langle x, w_t \rangle & \text{if } t > 1 \text{ and } x \neq \text{anchor}(S_j). \end{cases}
$$

Thus, by definition, $_xS_j$ is a path in G from x to $w_t$.  (See Lemma 3.3.6.)

Definition 2.3.8   Suppose C is an elementary circuit and $B \subseteq \text{nodes}(C)$, where $b = |B| \geq 2$.  Then B divides C into b edge-disjoint subpaths $P_1, \ldots, P_b$ such that $\bigcup_i \text{edges}(P_i) = \text{edges}(C)$.  Let these subpaths be called the sectors of C determined by B.  Thus each sector contains at least one edge of C, and each edge of C belongs to exactly one sector.

Definition 2.3.9   Suppose $G = (V, E)$, and $V' \subseteq V$.  A deficient node of V' is a node v in V' such that v is adjacent to at least one node in V-V' (equivalently, $\deg_{G[V']}(v) < \deg_G(v)$, where $G[V']$ denotes the section graph of V').

The $\underline{\text{Expansion}}$ $\underline{\text{Algorithm}}$ — $\underline{\text{Peripheral}}$ $\underline{\text{Version}}$ is most easily described in terms of the algorithm (L.1)-(L.2.2) given in Section 3.2.

Suppose that $G = (V, E)$ is a nonseparable graph and S is a (2, 1)-connected sequence for G. Let $S_1, S_2, \ldots, S_k$ denote the sequence of segments of S (see Definition 2.2.2). The $\underline{\text{Peripheral}}$ $\underline{\text{Version}}$ of the Expansion Algorithm is the same as the algorithm (L.1)-(L.2.2) except that the first paragraph of step (L.2) is replaced by the following:

(2.3.11)
$\left\{ \begin{array}{l} \end{array} \right.$ "If $j = k$, STOP (G is planar, and $G^* = G_k^*$ is a planar embedding of G, as will be proven in Theorem 3.3.17); otherwise, proceed. As will be proven in Theorem 3.3.17, $\text{Base}(S_{j+1}) \subseteq \text{nodes}(C_j)$ by construction of $C_j$, and $|\text{Base}(S_{j+1})| \geq 2$. If there exists a sector $P = \langle x_1, \ldots, x_q \rangle$ of $C_j$, determined by $\text{Base}(S_{j+1})$, such that $P \cup (_{x_1} S_{j+1}) \cup (_{x_q} S_{j+1})$ contains all deficient nodes of $S_1 \cup \ldots \cup S_{j+1}$, then proceed; otherwise, STOP (G is nonplanar). Let $e_0 = $ any edge in P (e.g., $(x_1, x_2)$). Let $j = j + 1$."

Note: If G is 3-connected, then the condition
"$P \cup (_{x_1} S_{j+1}) \cup (_{x_q} S_{j+1})$ contains all deficient nodes of $S_1 \cup \ldots \cup S_{j+1}$"
is equivalent to the following, computationally simpler condition (where $D_i$ denotes the set of deficient nodes of $S_1 \cup \ldots \cup S_i$, for any i):

"nodes (P) $\supseteq D_j \cap D_{j+1}$ (if $S_{j+1}$ is singular), or

nodes (P) $\supseteq D_j \cap D_{j+1}$ and anchor$(S_{j+1}) \in$ endpoints (P)

(if $S_{j+1}$ is nonsingular)."

To see this fact, observe that

$$D_{j+1} = D_{j+1} \cap (S_1 \cup \ldots \cup S_j \cup S_{j+1})$$

$$= (D_{j+1} \cap (S_1 \cup \ldots \cup S_j)) \cup (D_{j+1} \cap S_{j+1})$$

$$= (D_{j+1} \cap D_j) \cup (D_{j+1} \cap S_{j+1}) ;$$

hence, $P \cup (_{x_1} S_{j+1}) \cup (_{x_q} S_{j+1})$ contains the nodes of $D_{j+1}$ if and only if:
(i) nodes $(P) \supseteq D_{j+1} \cap D_j$, and (ii) $(_{x_1} S_{j+1}) \cup (_{x_q} S_{j+1})$ contains the
nodes of $D_{j+1} \cap S_{j+1}$. If $S_{j+1}$ is singular, then $D_{j+1} \cap S_{j+1}$ consists of
the single member of $S_{j+1}$ (by definition of (2, 1)-connected sequence),
so (ii) is necessarily satisfied and hence can be omitted. If
$S_{j+1} = \langle w_1, \ldots, w_t \rangle$ is nonsingular (i.e., $t > 1$), then $w_{t-1} \in D_{j+1} \cap S_{j+1}$
(since otherwise $w_{t-1}$ would be of degree 2 in G (by definition of $\underline{segment}$),
contradicting the 3-connectedness of G); hence, by Definition 2.3.7,
(ii) is satisfied if and only if $x_1$ or $x_q$ is equal to anchor $(S_{j+1})$; that is,
if and only if anchor $(S_{j+1}) \in$ endpoints $(P) = \{x_1, x_q\}$.

Chapter 3

## THEOREMS

### Section 1. OVERVIEW

The following sections establish the validity of the algorithms described in Chapter 2. Corresponding algorithms and proofs appear in correspondingly numbered subsections.

### Section 2. EXISTENCE OF (2, 1)-CONNECTED SEQUENCES

The purpose of this section is to prove constructively the existence of (2, 1)-connected sequences (and of primitive (2, 1)-connected sequences) for arbitrary 3-connected graphs. Section 2.1 establishes a number of helpful lemmas; Sections 2.2 and 2.3 prove the existence of (2, 1)-connected sequences and primitive (2, 1)-connected sequences, respectively.

The existence proofs contained in Sections 2.2 and 2.3 are constructive. These proofs are organized as rigorous validations of the algorithms (T.1)-(T.5) and (P.1)-(P.3) appearing in Chapter 2, Sections 2.2 and 2.3. Therefore, a good preparation for digesting these proofs is to review the intuitive discussion of the foregoing algorithms, as given in Chapter 2, Section 2.1.

### 2.1 Definitions and Lemmas

Before proceeding, we need to formalize the concept of "bridge," discussed in Chapter 2, Section 2.1.

3-2

Definition 3.2.1   A connected component of a graph $G = (V, E)$ is a subgraph of G which is connected, and which is not contained in a strictly larger connected subgraph of G. (Thus any two different connected components are node-disjoint and edge-disjoint.)

Definition 3.2.2   Suppose $G = (V, E)$ is a graph and $G' = (V', E')$ is a subgraph of G. Let $G[V-V']$ denote the section graph of $V-V'$ in G.  Let $C_1, \ldots, C_k$ ($k \geq 1$) denote the connected components of $G[V-V']$.  Let $d_1, \ldots, d_r$ ($r \geq 0$) denote the edges e of G having both endpoints in V', but not belonging to E'. (Thus when $G' = (V', E')$ is the section graph of V', $r = 0$.)

The bridges in G, relative to G', are the subgraphs

$$\overline{C}_i = C_i \cup \{\text{edges of G with one endpoint in } C_i \text{ and the other in V'}\} \ (i = 1, \ldots, k),$$

and $\overline{D}_j$ = subgraph consisting of $d_j$ ($j = 1, \ldots, r$).

If B is one of these bridges, then define $\text{int}(B) = \text{nodes}(B) \cap \text{nodes}(V-V')$ and $\overline{A}(B) = \text{nodes}(B) \cap \text{nodes}(V')$; $\overline{A}(B)$ is called the set of attachment points of B.

A singular bridge is a bridge containing exactly one edge; a bridge with more than one edge is called nonsingular.

Remarks on Definition 3.2.2   (i) By definition, the sets $\text{edges}(\overline{C}_1), \ldots, \text{edges}(\overline{C}_k), \text{edges}(\overline{D}_1), \ldots, \text{edges}(\overline{D}_r)$ are pairwise disjoint. Moreover, the union of these sets includes all the edges in E-E', since, for any edge e in E-E', either both endpoints of e belong to V' (in which case e belongs to a bridge $\overline{D}_j$), or at least one endpoint belongs to V-V' (in which case e belongs to a bridge $\overline{C}_i$).

(ii)   If $B = \overline{D}_j$, then $int(B) = \phi$ and $\overline{A}(B) = $ endpoints($d_j$). If $B = \overline{C}_i$, then $int(B) = nodes(C_i) \neq \phi$ and $\overline{A}(B) = $ nodes of $V'$ that are adjacent to at least one node in $nodes(C_i) = int(B)$. In any case, if $B$ and $B'$ are two distinct bridges, then $int(B) \cap int(B') = \phi$, since connected components are disjoint by definition.

(iii)   If $e$ is an edge with at least one endpoint in $int(B)$, then $B$ must be of the form $\overline{C}_i$, and hence $e$ belongs to edges($B$) (by definition of $\overline{C}_i$).

(iv)   For each node $v_0$ in $V-V'$, there is a unique bridge satisfying $v_0 \in int(B)$, namely, the bridge $B = \overline{C}_{i_0}$, where $C_{i_0}$ is the unique connected component containing $v_0$.

(v)   A nonsingular bridge must be of the form $\overline{C}_i$ defined above, since a bridge of the form $\overline{D}_j$ contains exactly one edge.

The following, somewhat lengthy sequence of lemmas prepares the way for the proof of Theorem 3.2.20 (viz., validity of algorithm (T.1)-(T.4.8)) in Section 2.2. The lemmas fall rather naturally into three groups, according to their utility. Lemmas 3.2.3 through 3.2.6 deal with properties of pre-chain sequences that are basic to justifying the algorithm (T.1)-(T.4.8). Lemmas 3.2.7 through 3.2.15 deal with fundamental properties of bridges. Lemmas 3.2.16 through 3.2.19 treat assorted technical details. We will give more detailed comments as we proceed.

The following lemma justifies the use of pre-chain sequences (as opposed to chain sequences) throughout algorithm (T.1)-(T.4.8). Recall the definition of "nodes(r)" in Definition 2.2.7.

**Lemma 3.2.3**   If r is a pre-chain sequence in a graph G, and nodes(r) = nodes(G), then r is a chain sequence in G.

**Proof:**   It suffices to show that, under the above hypothesis, condition (2) of Definition 2.2.7 implies condition (2) of Definition 2.2.6.  Indeed, suppose v, Z satisfy condition (2), Definition 2.2.7.  Then v is an endpoint of Z; let w denote the other endpoint.  By definition, $w \in \cup \{ \text{int}(R) \,|\, R > S \}$.  By condition (2.1) and the fact that nodes(r) = nodes(G), it follows that int(Z) = $\phi$; i.e., Z is a single edge.  Thus v is adjacent to w, so condition (2) of Definition 2.2.6 is satisfied. ∎

Lemmas 3.2.4 and 3.2.5 will be used to justify the initialization step (T.1) of the algorithm.

**Lemma 3.2.4**   Suppose G is a nonseparable graph.  Then there exists a chordless circuit in G.

**Proof:**   Let e = (v, w) be any edge in G.  Since G is nonseparable, there exists an elementary circuit containing e (see [Ber62], pg. 201).  Thus, there exists a path P in G from v to w, such that $e \notin P$.  Let $\overline{P}$ be such a path, of minimum length.  $\overline{P}$ is easily constructed by applying Moore's algorithm [Moo59] (also known as Lee's algorithm [Lee61]) to the graph G-edge e.

Then by construction, $e \notin \overline{P}$.  Moreover, $\overline{P}$ is a chain, because of its minimality.  Hence, $\overline{P} \cup \{e\}$ is a chordless circuit. ∎

**Lemma 3.2.5**   Suppose C = $\langle v_1, v_2, \ldots, v_{k+1} \rangle$ is a chordless circuit in a graph G (thus k ≥ 3 and $v_{k+1} = v_1$, by Definition 2.1.6).  Let r = $\langle R^1, R^2 \rangle$, where $R^1 = \langle v_1, v_2 \rangle$, $R^2 = \langle v_2, \ldots, v_{k+1} \rangle$.  Then r is a pre-chain sequence in G.

Proof: $R^1$ is a chain, since it has only one edge; $R^2$ is a chain, since any internal chord of $R^2$ would be a chord of C. $R^1$ and int($R^2$) are disjoint by construction. Condition (1) of Definition 2.2.7 is satisfied, since endpoints $(R^2) = \{v_2, v_{k+1}\} = \{v_2, v_1\} = $ nodes$(R^1)$. Finally, condition (2) of Definition 2.2.7 is satisfied vacuously. ∎

Lemma 3.2.6   If $r = \langle P, Q, \ldots, T \rangle$ is a pre-chain sequence, then nodes$(r)$ = nodes$(P) \cup$ int$(Q) \cup \ldots \cup$ int$(T)$.

Proof:   Suppose $v \in$ nodes$(r)$, and $v \notin$ nodes$(P)$. Let $R_0 = $ <u>first</u> path of r containing v. We will show that $v \in$ int$(R_0)$. Indeed, $v \notin$ endpoint$(R_0)$: since $R_0 \neq P$, Definition 2.2.7, condition (1) implies that endpoints $(R_0) \subseteq \cup \{$nodes$(S) \mid S < R_0\}$; thus, since $R_0$ is the <u>first</u> path containing v, $v \notin$ endpoints $(R_0)$. Hence the lemma. ∎

Lemmas 3.2.7 through 3.2.9 establish several basic properties of bridges.

Lemma 3.2.7   Suppose $G' = (V', E')$ is a subgraph of a graph $G = (V, E)$. Suppose $v \in V - V'$. Then there is a unique bridge B (in G, relative to G') such that $v \in$ int$(B)$.

Proof:   Let $G[V-V']$ denote the section graph of $V-V'$ in G. Let C denote the unique connected component of $G[V-V']$ containing v. Let $B = \overline{C}$, as defined in Definition 3.2.2. Then $v \in$ int$(B)$ by definition.

Moreover, B is unique, since for distinct bridges B, B', we have int(B) $\cap$ int(B') = $\phi$, as noted in Remark (ii) on Definition 3.2.2. ∎

Lemma 3.2.8   Suppose $G' = (V', E')$ is a subgraph of a graph $G = (V, E)$. If B is a <u>nonsingular</u> bridge (in G, relative to G'), then every node in

$\overline{A}(B)$ is adjacent to a node in $V-V'$.

Proof: The fact that B is nonsingular implies that B is of the form $\overline{C}_i$ (see Definition 3.2.2). Suppose $v \in \overline{A}(B)$. Then as noted in Remark (ii) on Definition 3.2.2, v is a node of $V'$ adjacent to a node in $C_i$. But nodes$(C_i) \subseteq V-V'$, by Definition 3.2.2. ∎

Lemma 3.2.9  Suppose $G' = (V', E')$ is a subgraph of a graph $G = (V, E)$. Suppose B is a bridge in G relative to $G'$. If int$(B) \neq \phi$ and $V' - \overline{A}(B) \neq \phi$, then $\overline{A}(B)$ is a separating set in G.

Proof:  By way of contradiction, suppose $\overline{A}(B)$ is not a separating set. Then the section graph of $V-\overline{A}(B)$ must be connected. By definition, int$(B) \subseteq V-V' \subseteq V-\overline{A}(B)$ (since $\overline{A}(B) \subseteq V'$); in addition, $V'-\overline{A}(B) \subseteq V - \overline{A}(B)$; since $V' \subseteq V$. Also, int$(B) \neq \phi$ and $V' - \overline{A}(B) \neq \phi$, by hypothesis. Hence (since $V-\overline{A}(B)$ is connected) for any $v \in$ int$(B) \neq \phi$, and $y \in V' - \overline{A}(B) \neq \phi$, there is a path in G from v to y, whose interior does not meet $\overline{A}(B)$. Let $\overline{P} = \langle v_1, \ldots, v_t \rangle$ $(v_1 = v,\ v_2 = y)$ be such a path, of minimal length.

We claim that int$(\overline{P}) = \phi$. Indeed, suppose int$(\overline{P}) \neq \phi$. Then $v_2 \in$ int$(\overline{P})$. Since $v_2$ is adjacent to $v_1 \in$ int$(B)$, edge $(v_1, v_2)$ must belong to B, as noted in Remark (iii) on Definition 3.2.2. Hence either $v_2 \in \overline{A}(B)$ or $v_2 \in$ int$(B)$. However, by construction of $\overline{P}$, $v_2 \notin \overline{A}(B)$, while if $v_2 \in$ int$(B)$, then path $\langle v_2, \ldots, v_t \rangle$ contradicts the minimality of $\overline{P}$. Therefore, it must be that int$(\overline{P}) = \phi$.

Thus $\overline{P}$ consists of the single edge $(v, y)$. Since $v \in$ int$(B)$, it follows that $(v, y) \in B$ (as noted in Remark (iii) on Definition 3.2.2). Hence $y \in \overline{A}(B)$ or $y \in$ int$(B)$. However, by definition, $y \notin \overline{A}(B)$; also,

y cannot belong to int(B), since $y \in V'$ by construction, and $int(B) \subseteq V-V'$. Thus we reach a contradiction.

Hence it must be that $\overline{A}(B)$ is a separating set. ∎

Lemma 3.2.10 and its corollary, Lemma 3.2.11, establish a basic property of pre-chain sequences in 3-connected graphs, namely, the existence of a chain whose interior is adjacent to "untreated" nodes. This property will be used to justify the existence of the path R described in step (T.2).

Lemma 3.2.10   Suppose $r = \langle P, Q, \ldots, T \rangle$ is a pre-chain sequence in a graph G, with nodes(G)-nodes(r) $\neq \phi$, and $|nodes(r)| \geq 3$. Suppose B is a bridge in G relative to the section graph of nodes(r). Then int(B) $\neq \phi$. Moreover, if G is 3-connected, then $|\overline{A}(B)| \geq 3$.

Proof:   Since B is a bridge relative to a section graph, B must be of the form $\overline{C}_i$, as described in Definition 3.2.2. Hence int(B) = nodes$(C_i) \neq \phi$.

Suppose G is 3-connected. By way of contradiction, suppose $|\overline{A}(B)| \leq 2$. Since $|nodes(r)| \geq 3$, we have nodes(r) $- \overline{A}(B) \neq \phi$. Hence by Lemma 3.2.9, $\overline{A}(B)$ is a separating set of G, contradicting the hypothesis that G is 3-connected. Hence it must be that $|\overline{A}(B)| \geq 3$. ∎

Lemma 3.2.11   Suppose $r = \langle P, Q, \ldots, T \rangle$ is a pre-chain sequence in a 3-connected graph G, with nodes(G)-nodes(r) $\neq \phi$ and $|nodes(r)| \geq 3$. Then there exists a chain S in r such that int(S) is adjacent to nodes(G)-nodes(r).

Proof:   Let v be any node in nodes(G)-nodes(r). Let B be the unique bridge (in G, relative to the section graph of nodes(r)) having $v \in int(B)$

(see Lemma 3.2.7). Then by Lemma 3.2.10, $|\overline{A}(B)| \geq 3$. In particular, B is nonsingular.

By Lemma 3.2.6, $nodes(r) = nodes(P) \cup int(Q) \cup \ldots \cup int(T)$. Hence, there must exist a path S in r such that $int(S)$ contains a node of $\overline{A}(B)$; otherwise, we would have $\overline{A}(B) \subseteq endpoints(P)$, contradicting $|\overline{A}(B)| \geq 3$. Let S be any such path and let w be a node in $int(S) \cap \overline{A}(B)$. Then by Lemma 3.2.8, w is adjacent to a node in $nodes(G) - nodes(r)$. Hence the lemma. ∎

Lemma 3.2.12 and its "converse," Lemma 3.2.13, establish the connection between bridges and the type of maximal subgraph "T" described in step (T.2): namely, (for suitably chosen T) $int(B) = nodes(T)$, and $\overline{A}(B) = A(T)$. This connection enables us to use bridges to prove several key properties of such subgraphs (for example, see proof of Lemma 3.2.15). The reason for employing the maximal subgraph concept is to avoid introducing the more complex notion of bridge into the formal statement of the algorithm (T.1)-(T.4.8).

Lemma 3.2.12   Suppose $r = \langle P, Q, \ldots \rangle$ is a pre-chain sequence in any graph G, with $nodes(G) - nodes(r) \neq \phi$. Suppose R is a path in r and B is a bridge (in G, relative to the section graph of $nodes(r)$) such that $\overline{A}(B) \cap int(R) \neq \phi$ and $int(B) \neq \phi$.

Let T be the section graph of $int(B)$. Then T is a subgraph of G that satisfies the following properties, and is maximal with respect to them:

    (a)   T is connected;

    (b)   $nodes(T) \subset nodes(G) - nodes(r)$;

    (c)   T is adjacent to $int(R)$.

Let $A(T) = \{w \in \text{nodes}(r) \mid w \text{ is adjacent to } T\}$. Then $\overline{A}(T) = A(B)$ and $\text{nodes}(T) = \text{int}(B)$.

Proof: First we will verify that T satisfies (a), (b), (c). Indeed, T is connected, by definition of a bridge (see Definition 3.2.2). Also, $T \subseteq \text{nodes}(G)\text{-nodes}(r)$, since $\text{int}(B) = \text{nodes}(B) \cap (V\text{-nodes}(r))$, by Definition 3.2.2. Finally, T is adjacent to $\text{int}(R)$: indeed, as noted in Remark (ii) on Definition 3.2.2, every node in $\overline{A}(B)$ is adjacent to a node in $\text{int}(B) = \text{nodes}(T)$, and by hypothesis, $\overline{A}(B)$ contains a node in $\text{int}(R)$.

Now we will show that T is maximal with respect to (a), (b), (c). Indeed, by way of contradiction, suppose $\overline{T}$ is a subgraph $\neq T$, which contains T and satisfies (a), (b), (c). Then $\text{nodes}(\overline{T})\text{-nodes}(T) \neq \phi$, since $\overline{T}$ is connected and contains T, at least one node of $\text{nodes}(\overline{T})\text{-nodes}(T)$ must be adjacent to $\text{nodes}(T)$; let v be such a node. Then v is adjacent to $\text{int}(B) = \text{nodes}(T)$ and $v \in \text{nodes}(G)\text{-nodes}(r)$ (by (b)), so $v \in \text{int}(B)$ (indeed, $\text{int}(B) \neq \phi$ by hypothesis, so by Definition 3.2.2, $\text{int}(B)$ is a connected component of the section graph of $\text{nodes}(G)\text{-nodes}(r)$). Hence $v \in T$, a contradiction. Thus it must be that T is maximal.

Finally, since $\text{int}(B) \neq \phi$, we have $\overline{A}(B) = \{w \in \text{nodes}(r) \mid w \text{ is adjacent to } \text{int}(B)\}$, as noted in Remark (ii) on Definition 3.2.2. Hence, since $\text{nodes}(T) = \text{int}(B)$ by definition of T, we have $A(T) = \overline{A}(B)$ by definition of $A(T)$. ∎

Lemma 3.2.13 Suppose $r = \langle P, Q, \ldots \rangle$ is a pre-chain sequence in any graph G, with $\text{nodes}(G)\text{-nodes}(r) \neq \phi$. Suppose R is a path in r and T is a subgraph of G which satisfies the following properties, and is maximal with respect to them:

(a)  T is connected;

(b)  nodes(T) $\subseteq$ nodes(G)-nodes(r);

(c)  T is adjacent to int(R).

Let v be any node in T, adjacent to int(R) (by (c), such a node exists).
Let B = the unique bridge (in B, relative to the section graph of
nodes(r)) having $v \in$ int(B) (see Lemma 3.2.7).

Then nodes(T) = int(B), T = section graph of int(B), and A(T) =$\overline{A}$(B),
where A(T) denotes $\{w \in$ nodes(r) $|$ w is adjacent to T$\}$.

If G is 3-connected, then $|A(T)| \geq 3$.

Proof:  Let C = the unique connected component of nodes(G)-nodes(r)
containing v; such a component exists because $v \in$ nodes(T) $\subseteq$ nodes(G)-
nodes(r). Thus, by the proof of Lemma 3.2.7, B = C $\cup$ $\{$edges of G with
one endpoint in C and the other in nodes(r)$\}$; thus int(B) = nodes(C) (see
Definition 3.2.2).

We now show that nodes(T)=nodes(C). Indeed, since $v \in$ nodes(T) and T
is maximal with respect to (a) and (b), it follows that T = C, by definition
of connected component; thus nodes(T) = nodes(C).

Furthermore, T = section graph of int(B), since int(B) = nodes(C)
as shown above, and T = C = section graph of nodes(C), by definition of
connected component (Definition 3.2.1).

Also, A(T) = $\overline{A}$(B), since A(T) = $\{w \in$ nodes(r) $|$ w is adjacent to T$\}$ =
$\{w \in$ nodes(r) $|$ w is adjacent to C$\}$ and as noted in Remark (ii) on
Definition 3.2.2, the last set is $\overline{A}$(B).

Finally, suppose G is 3-connected. By property (c), int(R) $\neq \phi$,
so $|$nodes(r)$| \geq 3$. Hence, by Lemma 3.2.10, $|A(T)| = |\overline{A}(B)| \geq 3$. ∎

Lemma 3.2.14 and its corollary, Lemma 3.2.15, establish the existence of the type of chain Q needed for augmentation (recall discussion in Chapter 2, Section 2.1).

<u>Lemma 3.2.14</u>  Suppose $G' = (V', E')$ is a subgraph of a graph $G = (V, E)$. Suppose B is a bridge (in G, relative to $G'$) with int(B) $\neq \phi$. Suppose $a_1, a_2$ are any two distinct nodes in $\overline{A}(B)$. Then there exists a chain Q from $a_1$ to $a_2$, such that

$$\phi \neq \text{int}(Q) \subseteq \text{int}(B).$$

<u>Proof:</u>  Since $a_i \in \overline{A}(B)$ and int(B) $\neq \phi$, there exists a node $v_i$ in int(B), adjacent to $a_i$ (i = 1, w) (see Remark (ii) on Definition 3.2.2). Since the section graph of int(B) is a connected component of $G[V-V']$ (see Definition 3.2.2), there exists a path $P = \langle v_1, \dots, v_2 \rangle$ from $v_1$ to $v_2$, with nodes(P) $\subseteq$ int(B). Hence, $\overline{P} = \langle a_1, v_1, \dots, v_2, a_2 \rangle$ is a path from $a_1$ to $a_2$, with

$$\phi \neq \text{int}(\overline{P}) \subseteq \text{int}(B).$$

Let Q be any such path from $a_1$ to $a_2$, of minimal length. Then by its minimality, Q must be a chain. ∎

<u>Lemma 3.2.15</u>  Suppose $r = \langle P, Q, \dots \rangle$ is a pre-chain sequence in G, with nodes(G)-nodes(r) $\neq \phi$. Suppose R is a chain in r, and T is a subgraph of G which satisfies the following properties and is maximal with respect to them:

(a)  T is connected;

(b)  nodes(T) $\subseteq$ nodes(G)-nodes(r);

(c)  T is adjacent to int(R).

Let A(T) = $\{w \in$ nodes(r) | w is adjacent to T$\}$. Suppose $a_1, a_2$ are distinct

nodes in $A(T)$. Then there exists a chain $Q$ in $G$ from $a_1$ to $a_2$ such that

$$\phi \neq \text{int}(Q) \subseteq \text{nodes}(T) .$$

Proof:   By Lemma 3.2.13, there exists a bridge $B$ (in $G$, relative to the section graph of nodes$(r)$ such that int$(B) \neq \phi$, nodes$(T) = \text{int}(B)$, and $A(T) = \overline{A}(B)$. Hence, by Lemma 3.2.14, there exists a chain $Q$ in $G$ from $a_1$ to $a_2$ such that

$$\phi \neq \text{int}(Q) \subseteq \text{int}(B) = \text{nodes}(T) . \blacksquare$$

As noted earlier, the following lemmas establish some basic technical results.

Lemma 3.2.16   Suppose $r = \langle P, Q, \dots, T \rangle$ is a pre-chain sequence in graph $G$, with nodes$(G)$-nodes$(r) \neq \phi$, and int$(P) = \phi$. Suppose $R$ is a path in $r$ such that int$(R)$ is adjacent to nodes$(G)$-nodes$(r)$, and suppose, moreover, that $R$ is the last such path in $r$. Then $s(R) \neq \phi$ and $s(R) \subseteq \text{int}(R)$ (see Definition 2.2.9).

Proof:   First, suppose $R \neq T$. By hypothesis, int$(R) \neq \phi$ and int$(P) = \phi$. Hence $R \neq P$, so by condition (2) of Definition 2.2.7, there exists a path $Z$ from a node $v \in \text{int}(R)$ to a node $w \in \bigcup \{\text{int}(S) \mid S > R\}$; suppose $w \in \text{int}(S')$, where $S' > R$. Now $Z$ must be a single edge; otherwise, int$(Z) \neq \phi$ and int$(Z) \subset \text{nodes}(G)$-nodes$(r)$ imply that int$(S')$ contains a node (namely, $w$) adjacent to nodes$(G)$-nodes$(r)$, contradicting the hypothesis on $R$ (since $S' > R$). Thus $v$ is adjacent to $w$, so $v \in s(R)$ by definition, whence $s(R) \neq \phi$. Also, since $R \neq T$ by assumption, we have $s(R) \subset \text{int}(R)$ by Definition 2.2.9.

Suppose $R = T$. Then $s(R) \neq \phi$ by Definition 2.2.9. Moreover, by

the hypothesis of the lemma, $\text{int}(R) \neq \phi$, so $s(R) = \{\text{next-to-last}$
node of $R\} \subseteq \text{int}(R)$. ∎

Lemma 3.2.17 and its corollary, Lemma 3.2.18, establish the
existence of the bridge $B'$ (equivalently, subgraph $T'$) that is needed
for the validation of step (T.4.7) (see discussion of Subcase 2.3 in
Chapter 2, Section 2.1).

<u>Lemma 3.2.17</u>   Suppose $G = (V, E)$ is a graph, $V' \subseteq V$, and $G' = (V', E')$
is the section graph of $V'$ in $G$. Suppose $P = \langle w_1, \ldots, w_t \rangle$ $(t \geq 3)$ is a
path in $G'$ such that

(1)   $V' - \text{nodes}(P) \neq \phi$ and

(2)   $w_i$ is not adjacent (relative to $G'$, or equivalently, $G$) to any
node in $V' - P$, for $1 < i < t$.

Suppose that, for any bridge $B$ (in $G$, relative to $G'$),
$\overline{A}(B) \cap \text{int}(P) \neq \phi$ implies $\overline{A}(B) \subseteq \text{nodes}(P)$. Then $\{w_1, w_t\}$ is a separating
set in $G$.

<u>Proof</u>:   By way of contradiction, suppose $\{w_1, w_t\}$ is not a separating
set in $G$. We will then show that there exists a bridge $B$ (in $G$, relative
to $G'$) with $\overline{A}(B) \cap \text{int}(P) \neq \phi$ and $\overline{A}(B) \not\subseteq \text{nodes}(P)$.

By assumption, then, the section graph of $V - \{w_1, w_t\}$ is connected.
Hence, if $v \in \text{int}(P)$ and $y \in V' - \text{nodes}(P)$, there exists a path $Q$ from $v$ to
$y$, with intermediate nodes in $V - \{w_1, w_t\}$. Moreover, by hypothesis,
$\text{int}(P) \neq \phi$ and $V' - \text{nodes}(P) \neq \phi$, so that at least one such path exists.
Let $\overline{Q}$ be any such path $Q$ of <u>minimum</u> length.

We will now show that $\text{int}(\overline{Q}) \cap V' = \phi$. Let $\overline{Q}$ be denoted by
$\langle q_1, \ldots, q_n \rangle$; we may assume $q_1 \in \text{int}(P)$, $q_n \in V' - \text{nodes}(P)$. By way of

contradiction, suppose $q_i \in V'$, for some $1 < i < n$. Then either $q_i$ does or does not belong to int(P). If $q_i \in$ int(P), then the path $\langle q_i, \ldots, q_n \rangle$ contradicts the minimality of $\overline{Q}$. If $q_i \notin$ int(P), then $q_i \notin$ nodes(P), since by hypothesis, int$(\overline{Q}) \cap \{w_1, w_t\} = \phi$; hence $q_i \in V' -$ nodes(P); and the path $\langle q_1, \ldots, q_i \rangle$ contradicts the minimality of $\overline{Q}$. Thus it must be that int$(\overline{Q}) \cap V' = \phi$.

Moreover, int$(\overline{Q}) \neq \phi$. Indeed, suppose int$(\overline{Q})$ were empty. Then $q_1 \in$ int(P) would be adjacent to $q_n \in V' -$ nodes(P), contradicting hypothesis (2) of the lemma. Thus int$(\overline{Q}) \neq \phi$. In particular, then, $q_2 \in V - V'$ by the preceding paragraph.

Let B be the unique bridge (see Lemma 3.2.7) in G, relative to G', containing $q_2$ in its interior. Since int(Q) $\subseteq V - V'$, it follows from the definition of <u>bridge</u> that int(Q) $\subseteq$ int(B), and $\{q_1, q_n\} \subseteq \overline{A}(B)$. Hence, since $q_1 \in$ int(P) and $q_n \notin$ nodes(P), it follows that $\overline{A}(B) \cap$ int(P) $\neq \phi$ and $\overline{A}(B) \not\subseteq$ nodes(P), contradicting the hypothesis of the lemma.

Thus it must be that $\{w_1, w_t\}$ is a separating set in G. ∎

<u>Lemma 3.2.18</u>  Suppose $r = \langle P, Q, \ldots, T \rangle$ is a pre-chain sequence in a 3-connected graph G, with nodes(G)-nodes(r) $\neq \phi$. Suppose R is the last chain in r whose interior is adjacent to nodes(G)-nodes(r) (by Lemma 3.2.11, such a chain exists). Suppose $K = \langle c_1, \ldots, c_2 \rangle$ is a subpath of R, with int$\langle c_1, \ldots, c_2 \rangle \neq \phi$ and int$\langle c_1, \ldots, c_2 \rangle \cap (s(R) \cup p(R)) = \phi$.

Then there exists a bridge B (in G, relative to the section graph of nodes(r)) such that

$$\overline{A}(B) \cap \text{int}(K) \neq \phi \text{ and}$$
$$\overline{A}(B) \not\subseteq \text{nodes}(K). \blacksquare$$

Proof: By way of contradiction, suppose that, for any bridge B (in G, relative to the section graph of nodes(r)), $\overline{A}(B) \cap \text{int}\langle c_1, \ldots, c_2 \rangle \neq \phi$ implies $\overline{A}(B) \subseteq \{c_1, \ldots, c_2\}$; we will show that $\{c_1, c_2\}$ is a separating set.

We will show that $V' = \text{nodes}(r)$, $K = \langle c_1, \ldots, c_2 \rangle$ satisfy the hypothesis of Lemma 3.2.17. Indeed, K is a path, and it contains three or more nodes, since $\text{int}\langle c_1, \ldots, c_2 \rangle \neq \phi$. Further, $V'-\text{nodes}(K) \neq \phi$, since $\text{nodes}(R) - \{c_1, \ldots, c_2\} \neq \phi$ (indeed, it suffices to show that endpoints $(R) \neq \{c_1, c_2\}$; this inequality is valid because $s(R) \neq \phi$ and $s(R) \subseteq \text{int}(R)$ (by Lemma 3.2.16) while $\text{int}\langle c_1, \ldots, c_2 \rangle \cap s(R) = \phi$ by hypothesis).

Moreover, $\text{int}(K)=\text{int}\langle c_1, \ldots, c_2 \rangle$ is not adjacent to $V'-\text{nodes}(K) = \text{nodes}(r)-\text{nodes}(K)$. Indeed, by Lemma 3.2.6, $\text{nodes}(r) = \text{nodes}(P) \cup \text{int}(Q) \cup \ldots \cup \text{int}(T)$; so suppose $v \in \text{int}\langle c_1, \ldots, c_2 \rangle$ is adjacent to $w \in \text{nodes}(r)-K = (\text{nodes}(P) \cup \text{int}(\ ) \cup \ldots \cup \text{int}(T)) - \{c_1, \ldots, c_2\}$. Then $w \notin R$, since if $w \in R$, then $w \notin \{c_1, \ldots, c_2\}$ implies that (v, w) is an internal chord of R. Suppose $w \in \text{nodes}(P)$; then $P \neq R$, so $P < R$ and $v \in p(R)$, a contradiction. Suppose $w \in \text{int}(R')$, where $R' \leq R$; then $R' \neq R$, so $R' < R$ and $v \in p(R)$, a contradiction. Suppose $w \in \text{int}(R')$, where $R' > R$; then $v \in s(R)$, a contradiction. Hence $\text{nodes}(r)$ and K satisfy the hypotheses of Lemma 3.2.17.

Thus, by Lemma 3.2.17, $\{c_1, c_2\}$ is a separating set in G, contradicting the 3-connectedness of G. Therefore, it must be that our initial assumption is invalid, and there exists a bridge B such that $\overline{A}(B) \cap \text{int}\langle c_1, \ldots, c_2 \rangle \neq \phi$ and $\overline{A}(B) \not\subseteq \{c_1, \ldots, c_2\}$. ∎

Lemma 3.2.19 establishes a result that is needed, for example, in

validating step (T.3.1) — namely, in proving that condition (2) in the definition of pre-chain sequence is satisfied.

Lemma 3.2.19   Suppose $G' = (V', E')$ is a subgraph of a graph $G = (V, E)$. Suppose B is a bridge in G relative to G', with $|\overline{A}(B)| \geqslant 3$. Suppose Q is a chain in G such that endpoints(Q) $\subseteq \overline{A}(B)$ and $\phi \neq$ int(Q) $\subseteq$ int(B).

Suppose y is any node in $\overline{A}(B)$-endpoints(Q). Then there exists a path Z from y to some node in int(Q), with int(Z) $\subseteq$ nodes(G)-(V' $\cup$ nodes(Q)).

Proof:   By hypothesis, int(Q) $\neq \phi$; let z be any node in int(Q). Let w be any node in int(B), adjacent to y (such a node exists, since $y \in \overline{A}(B)$ and int(B) $\neq \phi$ (by $|\overline{A}(B)| \geqslant 3$)). If w = z, then the lemma is satisfied. Otherwise, by definition of a bridge, there exists a path $P = \langle w, \ldots, z \rangle$ from w to z, with int(P) $\subseteq$ int(B). Let $\overline{P}$ denote the path $\langle y, w, \ldots, z \rangle$. Thus int($\overline{P}$) $\subseteq$ int(B).

Let $z_0$ denote the first node of $\overline{P}$ lying in int(Q) (such a $z_0$ exists, since $z \in$ int(Q)). Let Z be the subpath of $\overline{P}$ from y to $z_0$. Then by construction, int(Z) $\subseteq$ int(B) - int(Q). However, int(B) - int(Q) $\subseteq$ nodes(G) - (V' $\cup$ nodes(Q)). Indeed, $x \in$ int(B) implies $x \notin$ (V' $\cup$ endpoints(Q)), so $x \in$ int(B) - int(Q) implies $x \notin$ (V' $\cup$ nodes(Q)).

Hence the lemma. ∎

## 2.2  (2, 1)-Connected Sequences

The purpose of this subsection is to prove that algorithm (T.1)-
(T.5), applied to an arbitrary (planar or nonplanar) 3-connected graph
G, produces a (2, 1)-connected sequence with respect to G.  The proof
is divided into two parts:  first we show that algorithm (T.1)-(T.4.8)
produces a chain sequence containing all the nodes of the graph
(Theorem 3.2.20); then we show that a chain sequence produced by
algorithm (T.1)-(T.4.8) can be transformed into a (2, 1)-connected
sequence by step (T.5) (Theorem 3.2.21).

The proof of Theorem 3.2.20 is sufficiently complex to warrant
some introductory remarks.  Algorithm (T.1)-(T.4.8) consists of the
construction of a sequence r of paths in G (step (T.1)), followed by
repeated execution of steps (T.2)-(T.4.8) (until $nodes(G) = nodes(r)$).
We will show that step (T.1) produces a pre-chain sequence r.  Then,
assuming that $nodes(G) - nodes(r) \neq \phi$, we will show that each iteration
of steps (T.2)-(T.4.8) "augments" r, i.e., produces a pre-chain
sequence $\bar{r}$ with $nodes(\bar{r}) \supsetneq nodes(r)$.  The theorem immediately follows,
since in a finite number of iterations of (T.2)-(T.4.8), the algorithm will
produce a pre-chain sequence r with $nodes(G) = nodes(r)$, and any such
pre-chain sequence is necessarily a chain sequence (by Lemma 3.2.3).

To show that each iteration of (T.2)-(T.4.8) augments r, it suf-
fices to show that the following requirements are satisfied (recall that,
as summarized in the précis at the end of Section 2.1, Chapter 2, steps
(T.2)-(T.4.8) are divided into several cases).

(1)  the cases are exhaustive;

(2)  no "reduction" (from one case to another) produces circularity;

3-18

    (3)  for each case, the following conditions are satisfied:

        (3.1)  for any entity "chosen" from a given set, that set is in fact nonempty (i.e., such an entity exists);

        (3.2)  for any "reduction" to another case, the defining condition of that case is indeed satisfied;

        (3.3)  the pre-chain sequence r is "augmented," i.e., transformed into a pre-chain sequence $\bar{r}$ with $\text{nodes}(\bar{r}) \supsetneq \text{nodes}(r)$.

**Theorem 3.2.20**  Given a 3-connected graph G, there exists a chain sequence containing all the nodes of G. In fact, such a sequence can be constructed by algorithm (T.1)-(T.4.8) (Chapter 2, Section 2.2).

**Proof:**  By Lemma 3.2.3, it suffices to prove that there exists a pre-chain sequence containing all the nodes of G. We will prove that such a sequence can be constructed by algorithm (T.1)-(T.4.8).

    To start, we show that step (T.1) produces a pre-chain sequence r. Recall step (T.1):

        "Choose an arbitrary chordless circuit C in G. Suppose $C = \langle v_1, v_2, \ldots, v_k, v_{k+1} \rangle$ (where $v_{k+1} = v_k$). Let $R^1, R^2$ denote the paths $\langle v_1, v_2 \rangle$, $\langle v_2, \ldots, v_{k+1} \rangle$, respectively. Let $r = \langle R^1, R^2 \rangle$."

The fact that C is constructible follows from Lemma 3.2.4. Sequence r is a pre-chain sequence by Lemma 3.2.5.

    Now we assume that $r = \langle P, Q, \ldots \rangle$ is an "incomplete" pre-chain sequence (i.e., that $\text{nodes}(G) - \text{nodes}(r) \neq \phi$); to prove Theorem 3.2.30, it suffices to show that each iteration of steps (T.2)-(T.4.8) will "augment" r, i.e., produce a pre-chain sequence $\bar{r}$ with $\text{nodes}(\bar{r}) \supsetneq \text{nodes}(r)$.

As summarized in the précis at the end of Section 2.1 of Chapter 2, steps (T.2)-(T.4.8) are organized into cases and subcases. For the reader's convenience in following the proof, we repeat that précis here.

Précis of algorithm (T.1)-(T.4.8)

(T.1) Initialization.

(T.2) Test for completion; start of augmentation.

Choice of path R and subgraph T.

(T.3) Subdivision into Case 1 and Case 2 (for Case 2, go to T.4).

Case 1. $A(T) \not\subseteq nodes(R)$.

Subdivision into Subcases 1.1 and 1.2.

(T.3.1) Subcase 1.1 $|A(T) - int(R)| \geq 2$.

Construction (1), Figure 1.4 (Chapter 1).

(T.3.2) Return to T.2.

(T.3.3) Subcase 1.2 $|A(T) - int(R)| = 1$.

Construction (2), Figure 1.4.

(T.3.4) Test for "d-adjacency" problem.

(T.3.5) Treatment of "d-adjacency" problem.

(T.3.6) Return to T.2.

(T.4) Case 2. $A(T) \subseteq nodes(R)$.

Definition of $a_1, a_2$. Subdivision into Subcases 2.1, 2.2, and 2.3.

(T.4.1) Subcase 2.1 $int\langle a_1, \ldots, a_2 \rangle \cap s(R) \neq \phi$.

Construction (3), Figure 1.4.

(T.4.2) Return to T.2.

(T.4.3) Subcase 2.2 $int\langle a_1, \ldots, a_2 \rangle \cap s(R) = \phi$ and

$int\langle a_1, \ldots, a_2 \rangle \cap p(R) \neq \phi$.

Modification of r.

(T.4.4)  Test for "d-adjacency" problem.

(T.4.5)  Treatment of "d-adjacency" problem.

(T.4.6)  Reduction to Case 1.

(T.4.7)  <u>Subcase 2.3</u>    $\text{int}\langle a_1, \ldots, a_2 \rangle \cap s(R) = \phi$ and

$\text{int}\langle a_1, \ldots, a_2 \rangle \cap p(R) = \phi$.

Choice of subgraph T'.

(T.4.8)  Reduction to Case 1, Subcase 2.1, or

Subcase 2.2.

It suffices to prove that the requirements (1)-(3), discussed at the start of this subsection, are satisfied.

Let us show that requirement (1) (namely, exhaustiveness of the cases) is satisfied. First we show that the cases are well-defined, i.e., that their defining conditions (see above précis) are based on constructible entities. Specifically, we must prove the existence of R and T (step (T.2)) and the existence of $a_1, a_2$ (step (T.4)). The construction in step (T.2) runs as follows:

"Designate the nodes of r as <u>treated</u>. If all nodes of G are treated, go to T.5; otherwise, there exists at least one path S in r such that int(S) contains a node adjacent to an untreated node. Let R be the last such path in r. Construct any subgraph T that satisfies the following properties and is maximal with respect to them:

(a)  T is connected;

(b)  nodes(T) $\subseteq$ nodes(G) - nodes(r);

(c)  nodes(T) is adjacent to int(R).

Let $A(T) = \{w \in \text{nodes}(r) \mid w$ is adjacent to nodes(T)$\}$; thus $A(T) \cap \text{int}(R) \neq \phi$ by (c)."

By Lemma 3.2.11, at least one path S exists with int(S) adjacent to an untreated node ($|$nodes(r)$| \geq 3$, since the initial pre-chain sequence $\langle R^1, R^2 \rangle$ contains at least three nodes; see Lemma 3.2.5). Hence R (the last such path) exists. Given this R, a subgraph T such as that described in (T.2) is always constructible: let v be any untreated node adjacent to int(R) (by construction of R, such a node exists), let B be the unique bridge (with respect to the section graph of nodes(r)) containing v in its interior (see Lemma 3.2.7); thus $\bar{A}(B) \cap$ int(R) $\neq \phi$ and int(B) $\neq \phi$. Then by Lemma 3.2.12, T = (section graph of int(B)) satisfies the conditions in (T.2).

Furthermore, $a_1$ and $a_2$ exist (and, also, are distinct) because A(T) $\subseteq$ nodes(R) by choice of cases, $|$A(T)$| \geq 3$ by Lemma 3.2.13, and $a_1$ and $a_2$ are defined in step (T.4) as the first and last nodes (respectively) of R belonging to A(T).

Now let us show that the cases are exhaustive. As noted in the précis, the defining conditions of the cases are as follows:

Case 1.   A(T) $\not\subseteq$ nodes(R).

Subcase 1.1   $|$A(T) - int(R)$| \geq 2$.

Subcase 1.2   $|$A(T) - int(R)$| = 1$.

Case 2.   A(T) $\subseteq$ nodes(R).

Subcase 2.1   int$\langle a_1, \ldots, a_2 \rangle \cap$ s(R) $\neq \phi$.

Subcase 2.2   int $\langle a_1, \ldots, a_2 \rangle \cap$ s(R) $= \phi$ and
int $\langle a_1, \ldots, a_2 \rangle \cap$ p(R) $\neq \phi$.

Subcase 2.3   int$\langle a_1, \ldots, a_2 \rangle \cap$ s(R) $= \phi$ and
int $\langle a_1, \ldots, a_2 \rangle \cap$ p(R) $= \phi$.

It follows, by the law of the excluded middle, that the cases are

exhaustive (in particular, observe that in Case 1, since $A(T) \subseteq \text{nodes}(R)$, it must be that $A(T) \not\subseteq \text{int}(R)$; hence $|A(T) - \text{int}(R)| \geq 1$).

We now show that requirement (2) (namely, noncircularity of reductions from one case to another) is satisfied. As shown in the précis, the only such reductions are as follows:

Subcase 2.2:

Reduction to Case 1.

Subcase 2.3:

Reduction to Case 1, Subcase 2.1, or Subcase 2.2.

Since the only reductions made are reductions from one case to a preceding case, circularity cannot arise.

We now show that requirement (3.1) (existence of entities) is satisfied. Throughout algorithm (T.1)-(T.4.8), care was taken either to justify the existence of each new entity, or else to note that its existence would be proven later. There are precisely three situations in which existence (or nonemptiness) was in need of proof:

(i)   chain Q exists (in T.3.1, T.3.3, T.4.1);

(ii)   $s(R) \neq \phi$ (in T.3.3, T.4.3);

(iii)   subgraph T' exists (in T.4.7).

Existence of Q follows from Lemma 3.2.15. Let us show that $s(R) \neq \phi$. By assumption, we are given a pre-chain sequence $r = \langle P, Q, \ldots \rangle$ and a path R which is the last path of r whose interior is adjacent to an untreated node (i.e., a node of $\text{nodes}(G) - \text{nodes}(r)$). To show $s(R) \neq \phi$, it suffices, by Lemma 3.2.16, to show that $\text{int}(P) = \phi$. After the initial step T.1, the first path $R^1$ of r has empty interior, by construction.

Throughout the remainder of the algorithm, r is changed only by replacement of a path R, satisfying int(R) $\neq \phi$, or else by insertion of a path immediately before such an R. Since int(R) $\neq \phi$, it follows that $R \neq R^1$, so such replacement or insertion leaves $R^1$ as the first path in r. That is, the first path of P of r is always $R^1$, so int(P) = $\phi$. This completes the proof that s(R) $\neq \phi$. The existence of subgraph T' is guaranteed by Lemma 3.2.18 and Lemma 3.2.12.

Let us now show that requirement (3.2) is satisfied; this requirement states that, for any reduction to another case, the defining condition of that case is satisfied. The first possible reduction takes place in Subcase 2.2, step (T.4.6); it is a reduction to Case 1. We must show that, when this reduction takes place, r has been modified to a new pre-chain sequence whose associated path R (as specified in (T.2)) satisfies A(T) $\not\subseteq$ nodes(R) (the defining condition of Case 1). The fact that r (as modified in construction (T.4.3)-(T.4.5)) is a pre-chain sequence, follows directly from the construction: for example, properties (1) and (2) of Definition 2.2.7 are satisfied, by definition of $P_1 \oplus Q$, $P_2$, and $\overline{P}_1, \ldots, \overline{P}_k$; all paths in r are chains, since the only possible internal chords are those which arise in $P_1 \oplus Q$ because of adjacency to node d, and these chords are detected and eliminated by steps (T.4.4), (T.4.5). Moreover, A(T) $\not\subseteq$ nodes(R), since by construction, A(T) meets int($P_1$) (see step (T.4.3)).

The remaining reductions take place in Subcase 2.3, step (T.4.8). As noted in (T.4.7), R and the "new" T satisfy the specifications in (T.2). If A(T) $\not\subseteq$ nodes(R), then the defining condition of Case 1 is satisfied. If A(T) $\subseteq$ nodes(R), then by the construction in the preceding step (step T.4.7), we have int$\langle a_1, \ldots, a_2 \rangle \cap$ (s(R) $\cup$ p(R)) $\neq \phi$, where $a_1$ and $a_2$

denote the first and last nodes (respectively) of R belonging to A(T):

by condition (e), (step (T.4.7)) $\text{int}\langle a_1, \ldots, a_2 \rangle$ meets endpoints(K) -

endpoints(R) (recall that we are now assuming $A(T) \subseteq \text{nodes}(R)$); how-

ever, by maximality of K (see (T.4.7)), endpoints(K)-endpoints(R) is a

subset of $s(R) \cup p(R)$. Now, the condition $\text{int}\langle a_1, \ldots, a_2 \rangle \cap (s(R) \cup p(R))$

implies that the defining condition of Subcase 2.1 or Subcase 2.2 is

satisfied.

The only requirement remaining to be shown is (3.3). First we
will show that steps (T.2)-(T.4.8) preserve the pre-chain sequence
(pre-CS) property (Definition 2.2.7); afterwards, we will show that
these steps do indeed cause nodes(r) to be strictly enlarged. Since
modifications of r occur only in T.3.1, T.3.3, T.3.5, T.4.1, T.4.3,
and T.4.5, it suffices to check that the conditions of Definition 2.2.7
are preserved by these steps. In other words, it must be proven that
the modified r satisfies conditions (1) and (2) of Definition 2.2.7, as well
as the following two assumptions, stated in the definition: (3) each path
in r is a chain; (4) "disjointness": if $r = \langle P, Q, \ldots, T \rangle$, then nodes(P),
int(Q), ..., int(T) are pairwise disjoint sets. Let us procedd to verify
(1)-(4) for each of the steps T.3.1, T.3.3, T.3.5, T.4.1, T.4.3, T.4.5.
Below, we will use r (respectively, $\bar{r}$) to denote the sequence before
(respectively, after) modification.

<div align="center">Verification of condition (1):</div>

It suffices to consider only "new" paths, i.e., paths of $\bar{r}$ which are not
paths of r.

(T.3.1):     Endpoints(Q) = $\{b, d\} \subseteq A(T) - \text{int}(R)$ by construction. Hence,
            since Q is inserted immediately before R in r, it suffices to

show that $A(T) - int(R) \subseteq \cup \{nodes(S) | S \underset{r}{\leq} R\}$. By definition

of $T$, the nodes of $T$ are "untreated" (i.e., do not belong to

$nodes(r)$, so every node of $A(T)$ is adjacent to an untreated

node, by definition of $A(T)$. By definition, $R$ is the last

path of $r$ such that $int(R)$ is adjacent to an untreated node.

Since $nodes(r) = nodes(P) \cup int(Q) \cup \ldots \cup int(T)$ by

Lemma 3.2.6, and $A(T) \subseteq nodes(r)$, we have

$A(T) \subseteq nodes(P) \cup int(Q) \cup \ldots \cup int(R) = \cup \{nodes(S) | S \underset{r}{\leq} R\}$.

Hence $A(T) - int(R) \subseteq \cup \{nodes(S) | S \underset{r}{\leq} R\}$.

(T.3.3): As above, $d \in \cup \{nodes(S) | S \underset{r}{<} R\}$. Hence endpoints $(P_1 \oplus Q) \subseteq$

endpoints$(R) \cup \{d\}$ satisfies (1), since $P_1 \oplus Q$ is inserted in

the position of $R$. Also, $P_2$ satisfies (1), since

endpoints$(P_2) \subseteq$ endpoints$(R) \cup \{b\}$, $b \in nodes(P_1 \oplus Q)$, and $P_2$

follows $P_1 \oplus Q$.

(T.3.5): Each path in the sequence $\overline{P}_1 \oplus e_1, \ldots, \overline{P}_k \oplus e_k, P_{k+1} \oplus Q$

satisfies (1), since $d \in \cup \{nodes(S) | S \underset{r}{<} \{R\}$ as above and by

construction, endpoints $(\overline{P}_i \oplus e_i) = \{f_{i-1}, d\}$, $(i = 1, \ldots, k+1)$,

where $f_0 \in$ endpoints$(R)$, $e_{k+1} \overset{def}{=} Q$, and $f_{i-1} \in nodes(\overline{P}_{i-1})$

$(i = 2, \ldots, k+1)$.

(T.4.1): By construction, endpoints $(R_1 \oplus Q \oplus R_3)$ = endpoints $(R)$,

and endpoints $(R_2)$ = endpoints $(Q)$. Hence, since $R_1 \oplus Q \oplus R_3$

is inserted in place of $R$, and $R_2$ follows $R_1 \oplus Q \oplus R_3$,

condition (1) is satisfied.

(T.4.3): By construction, $d$ belongs to $nodes(P)$ for some $P \underset{r}{\leq} R$. Hence

condition (1) is satisfied, since endpoints $(P_1 \oplus Q) \subseteq$

$\{d\} \cup$ endpoints$(R)$, endpoints$(P_2) \subseteq nodes(Q) \cup$ endpoints$(R)$,

and $P_1 \oplus Q$, $P_2$ are inserted in place of R.

(T.4.5):     The proof here is exactly the same as for T.3.5 above.

## Verification of condition (2):

In verifying this condition, we must consider all paths of $\bar{r}$. We will begin by considering the "old" paths of $\bar{r}$ (i.e., paths of $\bar{r}$ which are also paths of r); then we will discuss the "new" paths of $\bar{r}$. First let us observe that by construction, for any nodes $v_1, v_2$ in nodes(r), we have $v_1 < v_2$ (rel r) $\Longrightarrow v_1 < v_2$ (rel $\bar{r}$) (where we use $v_1 < v_2$ (rel r) to denote the condition that there exist paths $R_1, R_2 \in r$ such that $R_1$ precedes $R_2$ in r, and $v_1 \in int(R_1)$, $v_2 \in int(R_2)$).

Now suppose S is any path in $\bar{r}$, other than the first, such that S also belongs to r. Because r is a pre-CS by hypothesis, there exists a node v in int(S), a node w in nodes(r), and a path Z in G from v to w such that int(Z) $\subseteq$ nodes(G) - nodes(r) and

$$v < w \text{ (rel r)} .$$

Suppose that int(Z) $\subseteq$ nodes(G) - nodes($\bar{r}$). Then since $v < w$ (rel r), we have $v < w$ (rel $\bar{r}$) as observed above, so v, Z satisfy condition (2) with respect to $\bar{r}$. On the other hand, suppose that int(Z) $\not\subseteq$ nodes(G)-nodes($\bar{r}$). Let $w_1$ be the first node in int(Z), belonging to nodes($\bar{r}$). Let $Z_1$ denote the subpath of Z from v to $w_1$. We will show that v, $Z_1$ satisfy condition (2) with respect to $\bar{r}$. Indeed, by definition of $Z_1$, we have int($Z_1$) $\subseteq$ nodes(G) - nodes($\bar{r}$). Hence it suffices to show that $v < w_1$(rel $\bar{r}$). Since int(Z) $\neq \phi$ and $v < w$ (rel r), S is not the last path of r whose interior is adjacent to an "untreated" node (i.e., a node of nodes(G) - nodes(r)); call this last path R. Thus S precedes R in r. Moreover, by construction, all new paths of $\bar{r}$ are inserted in r between R's immediate predecessor

and R's immediate successor — hence, after S. It follows that we must have $v < w_1$ (rel $\bar{r}$).

We now discuss the "new" paths of $\bar{r}$.

(T.3.1):    By construction (T.2), R follows in Q in $\bar{r}$, and A(T) meets int(R). Also $\phi \neq$ int(Q) $\subseteq$ nodes(T). Hence by Lemma 3.2.13 and Lemma 3.2.19, there exists a node in int(Q) and a path Z satisfying condition (2).

(T.3.3):    Since $b \in$ int($P_1 \oplus Q$) and b is an endpoint of $P_2$ (which follows $P_1 \oplus Q$ in $\bar{r}$), the edge Z of $P_2$ containing b will satisfy condition (2) (for $P_1 \oplus Q$), provided that int($P_2$) $\neq \phi$. Indeed, int($P_2$) $\neq \phi$, since $w \in$ int($P_2$), by construction.

Moreover, $P_2$ satisfies condition (2), since $w \in$ int($P_2$) and w is a member of s(R).

(T.3.5):    By construction, for $i = 1, \ldots, k$, $f_i$ belongs to int($\bar{P}_i \oplus e_i$), as well as to endpoints ($\bar{P}_{i+1} \oplus e_{i+1}$) (where we define $e_{k+1} = Q$); moreover, by construction, int($\bar{P}_{i+1} \oplus e_{i+1}$) $\neq \phi$ (in particular, int($e_{k+1}$) = int(Q) $\neq \phi$ by construction). Thus, for $i = 1, \ldots, k$, $\bar{P}_i \oplus e_i$ satisfies condition (2).

Now consider $\bar{P}_{k+1} \oplus Q$. If $f_k \neq b$, then b lies in int($\bar{P}_{k+1} \oplus Q$) and so condition (2) is satisfied, by the same argument as in case T.3.3 above. On the other hand, suppose $f_k = b$. Then ($\bar{P}_{k+1} \oplus Q$) = Q. However, by construction, $\phi \neq$ int(Q) $\subseteq$ nodes(T), and A(T) meets int($P_2$) (indeed, A(T) meets int($P_2$) in at least $\{a_1, a_2\} - \{b\}$). Hence by Lemma 3.2.13 and Lemma 3.2.19, there exists a path Z, such that int(Z) $\subseteq$ nodes(T), with one endpoint in int(Q) and

the other in $\text{int}(P_2)$. Thus condition (2) is satisfied by

$(P_{k+1} \oplus Q) = Q$, since $P_2$ follows $(\overline{P}_{k+1} \oplus Q) = Q$ in $\bar{r}$.

Lastly, $P_2$ satisfies condition (2), because $w \in \text{int}(P_2)$

by construction, and $w$ is a member of $s(R)$.

(T.4.1):   By hypothesis, $\{a_1, a_2\} \subseteq A(T) \subseteq R$. Suppose $\{a_1, a_2\} =$

endpoints(R). Then $A(T)$ meets $\text{int}(R_2)$, since $R_2 =$

$\langle a_1, \ldots, a_2 \rangle$ by definition, $\langle a_1, \ldots, a_2 \rangle = R$ by assumption,

$A(T) \subseteq R$ by hypothesis, and $|A(T)| \geq 3$ by Lemma 3.2.13.

Moreover, $\phi \neq \text{int}(Q) \subseteq \text{nodes}(T)$, by construction. Hence

by Lemma 3.2.13 and Lemma 3.2.19, there exists a path $Z$,

such that $\text{int}(Z) \subseteq \text{nodes}(T)$, with one endpoint in $\text{int}(Q)$ (and

therefore in $\text{int}(R_1 \oplus Q \oplus R_3)$) and the other in $\text{int}(R_2)$.

Thus condition (2) is satisfied by $R_1 \oplus Q \oplus R_3$, since $R_2$

follows $R_1 \oplus Q \oplus R_3$ in $\bar{r}$.

On the other hand, suppose $\{a_1, a_2\} \neq$ endpoints(R). Let

$a_i$ denote a member of $\{a_1, a_2\} -$ endpoints(R). Then $a_i \in$

$\text{int}(R_1 \oplus Q \oplus R_3)$. Moreover, $a_i \in$ endpoints$(R_2) = \{a_1, a_2\}$,

and $\text{int}(R_2) \neq \phi$ (indeed, $R_2 = \langle a_1, \ldots, a_2 \rangle$, and

$\text{int}\langle a_1, \ldots, a_2 \rangle \cap s(R) \neq \phi$, by hypothesis). Thus

$R_1 \oplus Q \oplus R_3$ satisfies condition (2), since $R_2$ follows

$R_1 \oplus Q \oplus R_3$ in $\bar{r}$.

Lastly, $R_2$ satisfies condition (2), since $R_2 = \langle a_1, \ldots, a_2 \rangle$

and $\text{int}\langle a_1, \ldots, a_2 \rangle$ meets $s(R)$ by hypothesis.

(T.4.3):   By construction, $b \in \text{int}(P_1 \oplus Q)$, and $b \in$ endpoints $(P_2)$.

Moreover, $\text{int}(P_2) \neq \phi$, since the node $w$ lies in $\text{int}(P_2)$ by

construction. Thus $P_1 \oplus Q$ satisfies condition (2), since $P_2$

follows $P_1 \oplus Q$ in $\bar{r}$.

$P_2$ satisfies condition (2), since by construction, $w \in \text{int}(P_2)$ and $w \in s(R)$.

(T.4.5):      By construction, for $i = 1, \ldots, k$, $f_i$ belongs to $\text{int}(\overline{P}_i \oplus e_i)$, as well as to endpoints $(\overline{P}_{i+1} \oplus e_{i+1})$(where we define $e_{k+1} = Q$; moreover, by construction, $\text{int}(\overline{P}_{i+1} \oplus e_{i+1}) \neq \phi$. Thus for $i = 1, \ldots, k$, $\overline{P}_i \oplus e_i$ satisfies condition (2).

     $\overline{P}_{k+1} \oplus Q$ satisfies condition (2), since $b \in \text{int}(\overline{P}_{k+1} \oplus Q)$, $b \in \text{endpoints}(P_2)$, $\text{int}(P_2) \neq \phi$ (since $w \in \text{int}(P_2)$ by construction), and $P_2$ follows $P_{k+1} \oplus Q$ in $\overline{r}$.

<div align="center">Verification of condition (3):</div>

It suffices to consider only "new" paths. In verifying this condition, some redundancy can be avoided if we notice some features common to all the steps (T.3.1), (T.3.3), (T.3.5), (T.4.1), (T.4.3), (T.4.5): namely, all paths, added as a result of these steps, are paths of the form $R_1 \oplus Q \oplus R_3$ (where one or both of $R_1, R_3$ may be empty), or $R_2$, where by hypothesis: $R_i$ are subpaths of a chain $R$; $R_1$ and $R_3$ are vertex-disjoint; and $Q$ is a chain (sometimes a single edge "$e_i$") with $\text{int}(Q) \cap \text{nodes}(r) = \phi$. Hence in particular, $R_1 \oplus Q \oplus R_3$ are <u>elementary</u> paths.

We need not concern ourselves with $R_2$, since it is necessarily a chain, inasmuch as it is a subpath of a chain. Similarly, if both $R_1$ and $R_3$ are empty, then $R_1 \oplus Q \oplus R_3 = Q$ is a chain, by hypothesis. So consider $R_1 \oplus Q \oplus R_3$, where at least one of $R_1, R_3$ is nonempty. We wish to prove, that by the simple hypotheses just described, it follows that: <u>if</u> e is an internal chord of $R_1 \oplus Q \oplus R_3$, then one endpoint of e <u>must</u> belong to $\text{int}(Q) \cup \text{endpoints} (R_1 \oplus Q \oplus R_3)$. Indeed, if not, then both

endpoints of e belong to $R_1 \cup R_3$-endpoints $(R_1 \uplus Q \oplus R_3)$, and the conclusion contradicts the hypothesis that R is a chain.

Let us now consider the individual steps:

(T.3.1):    By hypothesis, Q is a chain, and there is nothing to prove.

(T.3.3) and (T.3.5):    Since $P_2$ is a subpath of R, it is necessarily a chain. Let us therefore consider $P_1 \oplus Q$. By way of contradiction, suppose that $P_1 \oplus Q$ has an internal chord e. By the foregoing discussion, one endpoint of e must belong to $\text{int}(Q) \cup$ endpoints $(P_1 \oplus Q)$. Suppose v is such an endpoint, where $e = (v, w)$.

   (i)  Suppose $v \in \text{int}(Q)$. Then $w \notin \text{nodes}(Q)$, since Q is a chain, by hypothesis. Thus $w \in \text{nodes}(P_1)-\{b\}$. Since w is adjacent to $\text{int}(Q) \subseteq \text{nodes}(T)$, $w \in A(T)$ by definition. However, this conclusion contradicts the extremal condition defining the set $\{a_1, a_2\}$, to which b belongs — namely, $a_1$(resp. $a_2$) = first, (resp. last) node of R belonging to A(T).

   (ii) Suppose $v \in$ endpoints $(P_1 \oplus Q)$. Then either $v \in$ endpoints$(P_1)$ or $v \in$ endpoints$(Q)$.

      (iia)  If $v \in$ endpoints$(P_1)$, then $w \notin \text{nodes}(P_1)$, since $P_1$ is a subpath of R, and R is a chain; also, $w \notin Q$, by the "extremal argument" used above.

      (iib)  Thus $v \notin$ endpoints$(P_1)$, so $v \in$ endpoints$(Q)$. Thus $v = d$ (see construction in T.3.3, and hypothesis (ii)). Therefore, $w \notin \text{int}(Q)$, since Q is a chain. Hence $w \in \text{int}(P_1) \cup \{b\}$. In this case, however, the construction in T.3.5 is invoked, which explicitly

eliminates the existence of any such chords
(without introducing any internal chords in the
process — since no new nodes are introduced).

(T.4.1): Since $R_2$ is a subpath of R, it is necessarily a chain. Let
us therefore consider $R_1 \oplus Q \oplus R_3$. If $R_1$ and $R_3$ are empty,
then $R_1 \oplus Q \oplus R_3 = Q$ is a chain, by construction. If
exactly one of $R_1, R_3$ is empty, then precisely the same
argument as in (T.3.3)-(T.3.5) applies (with $P_1 = $ nonempty $R_i$),
up until (iib). At that point, we conclude as follows: in the
present instance, <u>both</u> endpoints of Q (viz., $a_1, a_2$) belong to
R, so "$v \in$ endpoints (Q)" is impossible, since neither R nor
Q has internal chords.

So suppose both $R_1$ and $R_3$ are nonempty. By way of
contradiction, suppose $R_1 \oplus Q \oplus R_3$ has an internal chord e.
As noted above, at least one endpoint v of e must belong to
$int(Q) \cup$ endpoints $(R_1 \oplus Q \oplus R_3)$. However, if $v \in int(Q)$, we
produce a contradiction, based on the fact that Q has no
internal chords, and $a_1, a_2$ satisfy the "extremal condition"
noted above. On the other hand, if $v \in$ endpoints $(R_1 \oplus Q \oplus R_3)$,
we produce a contradiction, based on the fact that R has no
internal chords, and $a_1, a_2$ satisfy the "extremal condition."
(Details are similar to those in the preceding arguments.)

(T.4.3) and (T.4.5): Since $P_2$ is a subpath of R, it is necessarily a
chain. Let us therefore consider $P_1 \oplus Q$. In the present
case, Q is the single edge (b, d). By way of contradiction,
suppose $P_1 \oplus Q$ has an internal chord e. As noted above,

at least one endpoint v of e must belong to int(Q) $\cup$ endpoints $(P_1 \oplus Q)$. Since int(Q) = $\phi$, v must therefore belong to endpoints $(P_1 \oplus Q)$. Since $P_1$ has no internal chords (being a subpath of R), and int(Q) = $\phi$, v cannot be the endpoint of $P_1 \oplus Q$ which is an endpoint of $P_1$; i.e., v can only be d. In this case, w (the other endpoint of e) belongs to int$(P_1)$, so the construction in T.4.5 is invoked, thereby explicitly eliminating any such chords (without introducing any internal chords in the process — since no new nodes are introduced).

Verification of condition (4) ("disjointness"):

The interiors of the chain(s) introduced at each step are pair-wise disjoint among themselves, by construction. Moreover, they are pair-wise disjoint with the other chains in the sequence, because they contain only nodes in the interior of the chain they replace, or else nodes not in nodes(r) at all.

To complete the proof of Theorem 3.2.20, it remains only to show that execution of steps (T.2)-(T.4.8) strictly enlarges nodes(r); i.e., that each of Subcases 1.1, 1.2, 2.1, 2.2, 2.3 strictly enlarges nodes(r). It suffices to consider only Subcases 1.1, 1.2, and 2.1, since Subcase 2.2 reduces to Subcase 1.1 or 1.2, and Subcase 2.3 reduces to Subcase 1.1, 1.2, 2.1 or 2.2 (as proven above); thus it suffices to consider only steps (T.3.1), (T.3.3), (T.4.1) (see précis at start of proof). In steps (T.3.1), (T.3.3), and (T.4.1), sequence r is replaced by a sequence $\bar{r}$ such that nodes$(\bar{r})$ = nodes$(r) \cup$ nodes(Q), where $\phi \neq$ int(Q) $\subset$ nodes(G)-nodes(r). Hence nodes$(\bar{r}) \supsetneq$ nodes(r).

This completes the proof of Theorem 3.2.20. ∎

We now wish to show that, given a chain sequenee $r = \langle F^0, F^1, \ldots, F^q \rangle$ for a 3-connected graph G, generated by algorithm (T.1)-(T.4.8), then step (T.5) produces a (2,1)-connected sequence. Theorem 3.2.21 is devoted to establishing this fact.

Before proceeding, however, let us observe that a chain sequence $r = \langle F^0, F^1, \ldots, F^q \rangle$ generated by algorithm (T.1)-(T.4.8) has the property that length $(F^0) = 1$. To prove this assertion, we note first that in the chain sequence $r = \langle R^1, R^2 \rangle$ constructed in step (T.1), chain $R^1$ is of length 1; hence it suffices to show that $R^1$ remains the first chain throughout the algorithm (T.2)-(T.4.8). Now the ehain R speeified in step (T.2) is not equal to $R^1$ (recall that R has nonempty interior, by construction), so that insertion of a path immediately before R (Subcase 1.1) or replacement of R by several paths (Subcases 1.2, 2.1, 2.2, 2.3), leaves $R^1$ as the first path in the sequence.

Thus, in order to prove that algorithm (T.1)-(T.5) generates a (2,1)-connected sequence for any graph, it suffices to prove the following theorem.

Theorem 3.2.21  Suppose $G = (V, E)$ is a 3-connected graph and $r = \langle F^0, F^1, \ldots, F^q \rangle$ is a "complete" chain sequence for G — i.e., a ehain sequence with nodes(r) = nodes(G). Suppose also that length $(F^0) = 1$.

Then step (T.5), applied to r, produees a (2,1)-eonnected sequenee with respeet to G.

<u>Proof</u>:    The proof falls into two main parts.  In the first part, we show

that we may assume r has several convenient, special properties.  In

the second part, we utilize these properties in proving that step (T.5)

produces a $(2, 1)$-connected sequence.

Let $F^j$ be denoted by $\langle v_j^1, \ldots, v_j^{t(j)}\rangle$ $(0 \leq j \leq q)$.  We may assume

here that $v_0^2 = v_1^1$, since by definition of a chain sequence, $\{v_1^1, v_1^{t(1)}\} =$

endpoints$(F_1) \subseteq$ nodes$(F_0) = \{v_0^1, v_0^2\}$ — and if $v_1^1 \neq v_0^2$, then $v_1^1 = v_0^1$, in

which case the first action of step (T.2) is to reverse path $F^1$, thereby

reversing its endpoints (note that this reversal preserves the chain

sequence property).

We will now show that, for purposes of proof, we may assume

$v_j^{t(j)-1} \in s(F^j)$ $(j \geq 1)$.  Indeed, let $v_j^{s(j)}$ denote the last node of

$\langle v_j^1, \ldots, v_j^{t(j)}\rangle$ belonging to $s(F^j)$ (by definition of a chain sequence,

$s(S) \neq \phi$ for every chain S except the first).  For each $j \geq 1$ such that

$s(j) \neq t(j) - 1$, proceed as follows.  Since G is 3-connected, $v_j^{s(j)+1}$ is

adjacent to at least one node in nodes$(F^0) \cup \ldots \cup$ nodes$(F^{k-1})$-endpoints$(F_j)$

(if not, $v_j^{s(j)+1}$ would be of degree 2, contradicting 3-connectedness of G);

let $d_j$ be any such node.  Replace $F^j$ by $\langle v_j^{t(j)}, v_j^{t(j)-1}, \ldots, v_j^{s(j)+1}, d_j\rangle = \overline{F}^j$

followed by $\langle v_j^1, \ldots, v_j^{s(j)+1}\rangle$.  This replacement corresponds precisely

to the operation described in step (T.4.3), where $v_j^{s(j)+1}$ corresponds to

b, $d_j$ corresponds to d, $\langle v_j^{s(j)+1}, d_j\rangle$ corresponds to Q, $v_j^{s(j)}$ corresponds

to w, $\langle v_j^{t(j)}, v_j^{t(j)-1}, \ldots, v_j^{s(j)+1}\rangle$ corresponds to $P_1$, $\langle v_j^1, v_j^2, \ldots, v_j^{s(j)+1}\rangle$

corresponds to $P_2$, and $\overline{F}^j$ corresponds to $P_1 \oplus Q$.

If any node in int$\langle v_j^{t(j)}, v_j^{t(j)-1}, \ldots, v_j^{s(j)+1}\rangle$ is adjacent to $d_j$, then

$\overline{F}^j$ is not a chain (recall the "d-adjacency problem" discussed in

Section 2.1 of Chapter 2); however, this problem can be remedied by

proceeding exactly as in step (T.4.5), breaking up $\overline{F}^j$ into several chains, and taking care to orient each such chain so that its last node is $d_j$ (since $d_j$ is the last node of $\overline{F}^j$, this "uniform" method of orienting chains enables us to assume for purposes of proof (see below) that step (T.4.5) is never applied.

Proceeding with these replacements ((T.4.3) and (T.4.5)), we obtain a new chain sequence $r' = \langle G^0, G^1, \ldots \rangle$; the fact that $r'$ is a chain sequence (or equivalently, a pre-chain sequence) follows directly from the validation of steps (T.4.3) and (T.4.5) in Theorem 3.2.20 (the fact that "R" $= F^j$ does not satisfy the property described in (T.2) is immaterial, because this special choice of R is used only to guarantee that $s(R) \neq \phi$). Most important, the construction of $r'$ guarantees that the next-to-last node of $G^i$ ($i \geq 1$) belongs to $s(G^i)$: indeed, observe that in the original sequence r, $v_j^{s(j)}$ (the next-to-last node of "$P_2$") belongs to $s(F^j)$ by definition, and $v_j^{s(j)+1}$ (the next-to-last node of "$P_1$") is adjacent to $v_j^{s(j)}$ (which belongs to int("$P_2$")). This property of next-to-last nodes is preserved by application of step (T.4.5), since the last node of "$\overline{P}_i$" (see step (T.4.5)) is identical to the first node of "$\overline{P}_{i+1}$."

We wish to show that by construction of $r'$, step (T.5) produces the same sequence, when applied to $r'$, as it does when applied to r. In verifying this fact, we may assume that step (T.4.5) is never applied, since $\overline{P}_1 \circ \overline{P}_2 \circ \ldots \circ \overline{P}_k$ (see T.4.5) equals $P_1$ by construction — so the effect of step (T.5) on $\overline{P}_1 \oplus e_1, \ldots, \overline{P}_k \oplus e_k, \overline{P}_{k+1} \oplus Q$ (see T.4.5) is the same as the effect of step (T.5) on $P_1 \oplus Q$ (recall that we specified above that all these paths have $d_j$ as their last node, thus avoiding the possibility of difficulties due to accidentally conflicting orientations).

We may assume, then, that $r'$ is as follows:

$$r' = \langle G^0, G^1, \ldots, G^{2q} \rangle$$

$$= \langle \langle v_0^1, v_0^2 \rangle, \langle v_1^{t(1)}, v_1^{t(1)-1}, \ldots, v_1^{s(1)+1}, d_1 \rangle, \langle v_1^1, \ldots, v_1^{s(1)+1} \rangle,$$

$$\langle v_2^{t(2)}, v_2^{t(2)-1}, \ldots, v_2^{s(2)+1}, d_2 \rangle, \langle v_2^1, \ldots, v_2^{s(2)+1} \rangle,$$

$$\vdots$$

$$\langle v_q^{t(q)}, v_q^{t(q)-1}, \ldots, v_q^{s(q)+1}, d_q \rangle, \langle v_q^1, \ldots, v_q^{s(q)+1} \rangle \rangle.$$

Because the next-to-last node of $G^i$ belongs to $s(G^i)$ $(i \geqslant 1)$, as noted above, the result of applying (T.5) to $r'$ is the sequence $G^0 \circ \hat{G}^1 \circ \ldots \circ \hat{G}^{2q}$ (where $\hat{G}^i$ denotes the subsequence of $G^i$ obtained by deleting the first and last nodes of $G^i$), which is identical, by definition, to the sequence $F^0 \circ F_{(2)}^1 \circ F_{(1)}^1 \circ \ldots \circ F_{(2)}^q \circ F_{(1)}^q$ obtained by applying step (T.5) to r.

Thus, we may proceed with the proof, while assuming that $v_j^{t(j)-1} = v_j^{s(j)} \in s(F^j)$ $(j \geqslant 1)$. In this case, the sequence T produced by step (T.5) is simply $T = F^0 \circ \hat{F}^1 \circ \ldots \circ \hat{F}^q$. First we note that T is a linear ordering of the nodes of V. Indeed, since r is complete, and since nodes($F^0$), int($F^1$), ..., int($F^q$) are disjoint, T contains each node of v exactly once.

Suppose we denote T by $\langle t_1, \ldots, t_k \rangle$. To show that T is a (2,1)-connected sequence, it suffices to show that

(a)   $(\forall_j, 1 < j < k)$  $G^{\geqslant j} (\overset{\text{def}}{=}$ section graph of $\{t_i \mid i \geqslant j\})$ is connected;

(b)   $(\forall_j, 1 < j < k)$  the following condition is satisfied:

(b.1)   $G^{\leqslant j}$ is nonseparable, or $t_j$ is of degree 1 in $G^{\leqslant j}$ and is adjacent to $t_{j+1}$.

Proof of (a).

To show that a graph H is connected, it suffices us to show that $(\exists\, x \in H)(\forall\, y \in H)$ there is a path $P(y, x)$ in H from y to x. We will show that $(\forall\, t \in G^{\geq j})$ there is a path $P(t, t_k)$ in $G^{\geq j}$ from t to $t_k$. However, to establish the existence of such a path (for all j), it suffices to show

(a.1)     $(\forall\, t \in T,\, t \neq t_k)(\exists\, t' \in T)$ such that the following condition is satisfied:

       (a.1.1)     $t < t'$ (relative to the ordering $t_1 < \ldots < t_k$) and $t, t'$ are adjacent.

Proof of (a.1):

(i)     If $t = v_0^1$, then $t' = v_0^2$ is adjacent to t, by definition of CS, and $v_0^1 < v_0^2$.

(ii)     If $t = v_0^2$, then $\exists\, t' \in \text{int}(F^1)$ adjacent to t, since $\text{endpoints}(F^2) = \{v_0^1, v_0^2\}$ by definition of CS; by definition of $T, t < t'$.

(iii)     Suppose $t = v_j^i \in \text{int}(F^j) - \{v_j^{t(j)-1}\}$, $1 \leq j \leq k$. (Recall that $F^j$ is a path.) Then $t' = v_j^{i+1} \in \text{int}(F^j)$ satisfies (a.1.1).

(iv)     Suppose $t = v_j^{t(j)-1}$, $1 \leq j \leq k$.
Then $v_j^{t(j)-1} \in s(F^j)$ (see beginning of proof), so $\exists\, v \in \text{int}(F^{j+1}) \cup \ldots \cup \text{int}(F^k)$ adjacent to $v_j^{t(j)-1}$.
Thus $t' = v$ satisfies (a.1.1).

Since (i)-(iv) exhausts all possibilities, (a.1) is satisfied.

Proof of (b).

In establishing (b), we will make use of the following property of nonseparable graphs, which follows directly from the definition:

(3.2.22) $\begin{cases} \text{Suppose } G_1 = (V_1, E_1) \text{ is a nonseparable graph.} \\ \text{Suppose } P \text{ is an elementary path not in } G_1 \text{ (of} \\ \text{length} \geq 1) \text{ with endpoints in } V_1 \text{ and int}(P) \cap V_1 = \phi. \\ \text{Let } G_2 = (V_2, E_2) \text{ where } V_2 = \text{nodes}(P), \ E_2 = \text{edges}(P). \\ \text{Then the graph } G_3 = G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2) \text{ is} \\ \text{nonseparable.} \end{cases}$

Recall that $T = F^0 \circ \hat{F}^1 \circ \ldots \circ \hat{F}^q$; thus we see that, to show (b), it suffices to establish (b.1) for all $t_j$ of the following form:

(i)     $t_j = v_0^2$;

(ii)    $t_j = v_i^{t(i)-1}$, $1 \leq i < q$;

(iii)   $t_j = v_i^u$, $1 \leq i \leq q$, $2 \leq u < t(i)-1$.

(i)     Suppose $t_j = v_0^2$. Then $j = 2$ and $t_2$ is of degree 1 in $G^{\leq 2}$, since $t_1 = v_0^1$, $t_2 = v_0^2$ are adjacent, while $t_2$ is adjacent to $t_3 = v_1^2$, since $v_0^2 = v_1^1$ (see start of proof).

(ii)    Suppose $t_j = v_i^{t(i)-1}$, $1 \leq i < q$. Then we will show that $G^{\leq j}$ is nonseparable. Note that by definition of $t_j$, $G^{\leq j} = F^0 \cup F^1 \cup \ldots \cup F^i \cup E^i$, where $E^i$ is a set of edges with endpoints in $F^0 \cup F^1 \cup \ldots \cup F^i$. Hence it suffices to show that $F^0 \cup F^1 \cup \ldots \cup F^i$ is non-separable ($1 \leq i < q$), since by definition of nonseparability, the addition of edges between nodes of a nonseparable graph preserves nonseparability.

We proceed by induction on i. $F^0 \cup F^1$ is non-separable, since it is just an elementary circuit (by Definition 2.2.6). Suppose $F^0 \cup F^1 \cup \ldots \cup F^{i_0}$ is

nonseparable, $(1 \leq i_0 < q-1)$. Then

$F^0 \cup F^1 \cup \ldots \cup F^{i_0} \cup F^{i_0+1}$ is nonseparable by

(3.2.22), since $F^{i_0+1}$ is an elementary path with end-

points in $F^0 \cup F^1 \cup \ldots \cup F^{i_0}$ (see Definition 2.2.6)

and $\text{int}(F^{i_0+1}) \cap \text{nodes}(F^0) \cup \ldots \cup \text{nodes}(F^{i_0})) =$

$\text{int}(F^{i_0+1}) \cap \text{nodes}(F^0) \cup \text{int}(F^1) \cup \ldots \cup \text{int}(F^{i_0})) = \phi$

by the disjointness condition of Definition 2.2.6.

(iii)  Suppose $t_j = v_i^u$, where $1 \leq i \leq q$, $2 \leq u < t(i) - 1$.

We note that $t_j$ is of degree $\geq 1$ in $G^{\leq j}$: indeed,

$t_j = v_i^u$ is adjacent to $v_i^{u-1}$, and if $u > 2$ then

$v_i^{u-1} = t_{j-1} \in G^{\leq j}$, while if $u = 2$, then $v_i^{u-1} = v_i^1$

belongs to $\text{nodes}(F^0) \cup \text{int}(F^1) \cup \ldots \cup \text{int}(F^{i-1}) \subseteq G^{\leq j}$.

Case 1  Suppose $t_j$ is of degree 1 in $G^{\leq j}$.

Note that $t_j$ is adjacent to $t_{j+1}$. Indeed, since

$t_j = v_i^u$ (where $u < t(i)-1$), we have $t_{j+1} = v_i^{u+1}$ by

definition of T. But $F^i = \langle v_i^1, v_i^2, \ldots, v_i^{t(i)} \rangle$ is a

path, so $v_i^u$, $v_i^{u+1}$ are adjacent.

Thus (b.1) is satisfied.

Case 2  Suppose $t_j$ is of degree $>1$ in $G^{\leq j}$. We will

show that $G^{\leq j}$ is nonseparable. By hypothesis

$t_j = v_i^u$, $i \geq 1$, $u \geq 2$. Thus $t_{j-1} = v_i^{u-1}$, and $t_j$,

$t_{j-1}$ are adjacent. Hence, in order that $t_j$ be of

degree $>1$ in $G^{\leq j}$, it must be that $\exists w \in$

$\text{nodes}(F^0) \cup \text{int}(F^1) \cup \ldots \cup \text{int}(F^{i-1}) \cup \{v_i^2, \ldots, v_i^{u-2}\}$

$= \text{nodes}(G^{<j-1})$

such that $w$ is adjacent to $t_j$.

However, $w \notin \{v_i^2, \ldots, v_i^{u-2}\}$, since otherwise $(t_j, w)$ would be an internal chord in $F^i$. Similarly, $w \neq v_i^1$. Thus $P = \langle v_i^1, v_i^2, \ldots, v_i^u, w \rangle$ is an elementary path. Also, since $w \in$ nodes$(F^0) \cup$ int$(F^1) \cup \ldots \cup$ int$(F^{i-1})$, $P$ is a path with endpoints in nodes$(F^0) \cup$ int$(F^1) \cup \ldots \cup$ int$(F^{i-1})$, and int$(P) \cap (F^0 \cup$ int$(F^1) \cup \ldots \cup$ int$(F^{i-1})) = \phi$. Hence $F^0 \cup F^1 \cup \ldots \cup F^{i-1} \cup P$ is nonseparable by (3.2.22), since $F^0 \cup F^1 \cup \ldots \cup F^{i-1}$ is nonseparable by (i), (ii).

However, $G^{\leq j} = F^0 \cup F^p \cup \ldots \cup F^{i-1} \cup P \cup E$, where $E$ is a set of edges having endpoints in $F^0 \cup F^1 \cup \ldots \cup F^{i-1} \cup P$, so $G^{\leq j}$ is nonseparable by (3.2.22).

This completes the proof of Theorem 3.2.21. ∎

## 2.3  Primitive (2, 1)-Connected Sequences

The purpose of this subsection is to prove the following theorem.

Theorem 3.2.23  Suppose G is a 3-connected graph.  Then the result of applying algorithm (P.1)-(P.3) (Chapter 2, Section 2.3) is a primitive (2, 1)-connected sequence with respect to G.

Proof:   Let $S = \langle v_1, v_2, \ldots \rangle$ denote the (2, 1)-connected sequence generated by (P.1) (see Theorems 3.2.20 and 3.2.21).  First, we show that $C = \langle v_1, \ldots, v_{i_0}, v_1 \rangle$ (as described in P.2)) is a chordless circuit.  By construction, C is a circuit, since by definition, $v_{i_0}$ is the first node of a (2, 1)-connected sequence adjacent to two or more preceding nodes — so $v_{i_0}$ must be adjacent to $v_1$ (otherwise the 2-connected condition of Definition 2.2.1 would be violated).  Moreover, C is chordless, because by proof of Theorem 3.2.21, we may assume $v_1, \ldots, v_{i_0}$ is a subpath of $F^0 \circ \hat{F}^1$ (recall definition in proof) and $F^1$ is a chain.

Thus, step (P.3) can be carried out, since step (T.1) applies to any chordless circuit.  By Theorems 3.2.20 and 3.2.21, the result will be a (2, 1)-connected sequence $\overline{S} = \langle \overline{v}_1, \overline{v}_2, \ldots \rangle$.  We wish to show that $\overline{S}$ is a primitive (2, 1)-connected sequence.  To do so, we will prove that the circuit C is a primitive circuit through $(\overline{v}_1, \overline{v}_2)$ and the last node of $\overline{S}$.  To begin, the chordless circuit C is not a separating set, inasmuch as the section graph of (nodes(G)-nodes(C) = $\{v_{i_0+1}, v_{i_0+2}, \ldots\}$) is connected, by definition of a (2, 1)-connected sequence.

Next we wish to show that C contains the edge $(\overline{v}_1, \overline{v}_2)$.  By definition, step (T.3) consists of the application of algorithm (T.1)-(T.5) beginning with the circuit $C = \langle v_1, \ldots, v_{i_0}, v_1 \rangle$.  Hence the initial

3-42

pre-chain sequence constructed in (T.1) consists of $r = \langle R^1, R^2 \rangle$, where $R^1 = \langle v_1, v_2 \rangle$ and $R^2 = \langle v_2, \ldots, v_{i_0}, v_1 \rangle$. As proven in the remarks preceding Theorem 3.2.21, $R^1$ remains the first path in r, throughout the application of steps (T.2)-(T.4.8). Hence, upon execution of step (T.5), $v_1, v_2$ become the first two nodes in $\overline{S}$; i.e., $v_1 = \overline{v}_1$, $v_2 = \overline{v}_2$. Thus circuit C contains the edge $(\overline{v}_1, \overline{v}_2) = (v_1, v_2)$.

To complete the proof, we must show that C contains the last node of $\overline{S}$. Suppose that $r = \langle F^0, F^1, \ldots, F^q \rangle$ is the (2, 1)-connected sequence produced in step (P.3) by the application of algorithm (T.1)-(T.4.8). Let $F^i = \langle v_i^1, \ldots, v_i^{t(i)} \rangle$, $0 \leq i \leq q$. By definition, $s(F^q) = \{ v_q^{t(q)-1} \}$; hence $v^{s(q)} = v^{t(i)-1}$, since $v^{s(q)}$ is defined (see (T.5)) as the last node in $F^q$ belonging to $s(F^q)$. It follows, by the construction in (T.5), that $v_q^{s(q)} = v_q^{t(q)-1}$ is the last node of $\overline{S}$.

Thus, it suffices to show that $v^{t(q)-1} = v_{i_0}$, since C contains $v_{i_0}$. First, we observe that $v_{i_0}$ is the next-to-last node of $R^2 = \langle v_2, \ldots, v_{i_0}, v_1 \rangle$ (see above) — where $r = \langle R^1, R^2 \rangle$ is the pre-chain sequence initially constructed in step (P.3). Hence, it suffices to show that, throughout the augmentation of r (steps (T.2)-(T.4.8)) the following property is preserved:

(3.2.24)... node $v_{i_0}$ is the next-to-last node of the last path in r.

(Recall that by definition, $v_q^{t(q)-1}$ is the next-to-last node of the last path in $r = \langle F^0, F^1, \ldots, F^q \rangle$.)

We must consider each step of (T.2)-(T.4.8) which alters the structure of r: namely, steps T.3.1, T.3.3, T.3.5, T.4.1, T.4.3, and T.4.5. It suffices to assume that the path R specified in T.2 is the last path of the uncompleted pre-chain sequence — for if R is a path other than the last, then the last path remains unaltered and property (3.2.24)

is necessarily preserved. Let us consider each step in turn. Step T.3.1 preserves (3.2.24), since in this step, the path "Q" is inserted before R, and R remains unchanged. Step T.3.3 preserves (3.2.24), since in this step, "w" must equal $v_q^{t(q)-1}$ (the sole member of $s(R)$), so the next-to-last node of "$P_2$" is $v_q^{t(q)-1}$ (recall that, as noted in (P.3), the nodes of "$P_2$" appear in the same order as they do in R) — and "$P_2$" becomes the last path of r. Step T.3.5 preserves (3.2.24), since this step modifies only path "$P_1$," which precedes the path "$P_2$" just mentioned.

Step T.4.1 preserves (3.2.24), since in this step "w" must equal $v_q^{t(q)-1}$ (the sole member of $s(R)$), so the next-to-last member of "$R_2$" is $v_q^{t(q)-1}$ (recall that, as noted in (P.3), the nodes of "$R_2$" appear in the same order as they do in R) — and "$R_2$" becomes the last path of r. Step T.4.3 and step T.4.5 preserve (3.2.24) for the same reasons as discussed for steps T.3.3 and T.4.5, respectively.

This completes the proof of Theorem 3.2.23. ∎

## Section 3. VALIDITY OF EXPANSION ALGORITHM

An intuitive discussion of this section appears in Chapter 2,

Section 3.1. Throughout the sequel, a uniform notation ("$[G']$") will be

used to denote section graphs (see Definition 2.1.9). If G is planar and $G^*$

is a planar embedding of G, then the restriction of $G^*$ to $G[V']$ will be de-

noted by $G^*[V']$. In general, a geometric entity corresponding to a graph-

theoretic entity "x" will be denoted by "$x^*$." For notational convenience,

a segment $S_j$ (see Definition 2.2.2) will be identified with the set of nodes

it contains; thus, set-theoretic expressions such as "$\{v, w\} \subseteq S_j$," etc.,

will be used freely.

### 3.1 Lemmas

The succeeding lemmas will be used in proving the validity of the

Expansion Algorithm/Linear Version (Theorem 3.3.12). The lemmas

fall into two groups: Lemmas 3.3.1 through 3.3.7a,b establish the con-

nection between linear expansion (Definition 2.3.6) and (the segments of)

(2,1)-connected sequences; Lemmas 3.3.8 through 3.3.11 contain techni-

cal details needed in the proof of Theorem 3.3.12. We will give

additional comments as the lemmas appear.

Lemmas 3.3.1 and 3.3.2 establish the basic background for linear

expansion, namely, the existence of elementary circuits.

Lemma 3.3.1   Suppose H is a nonseparable graph and S is a (2, 1)-

connected sequence for H. Let $S_1, \ldots, S_k$ denote the sequence of seg-

ments of S, and let $H_j$ be the section graph of $S_1 \cup \ldots \cup S_j$. Then $H_j$

is nonseparable (j = 1, \ldots, k).

Proof: Suppose $S_j = \langle v_1, \ldots, v_t \rangle$. Since $S_j$ is a segment, $v_t$ must be adjacent to two or more nodes of S preceding $v_t$. Hence, by definition of (2, 1)-connectedness, the section graph of the nodes of S up to and including $v_t$ (that is, $H_j$) must be nonseparable. ∎

The following lemma is a well-known result due to Whitney [Whi32].

Lemma 3.3.2 Suppose H is a planar, nonseparable graph and $H^*$ is a planar embedding of H. Then the underlying graph of the boundary of each face of $H^*$ is an elementary circuit (and is therefore represented by a closed Jordan curve in $H^*$).

Proof: See [Whi32], pages 339 ff., and Theorem 19 (page 350). ∎

Lemmas 3.3.3 through 3.3.6 enable us to prove formally that each of the sets $F_j$ (j > 1) discussed in Chapter 2, Section 3.1 actually has the "star-like" structure described in Chapter 2, Section 3.1.

Lemma 3.3.3 Suppose H is a planar, nonseparable graph and $S = \langle w_1, \ldots, w_n \rangle$ is a (2, 1)-connected sequence for H, with segments $S_1, \ldots, S_k$. Suppose $H^*$ is a planar embedding of H such that $w_n^*$ lies on the periphery of $H^*$. Let $H_j^* = H^*[S_1 \cup \ldots \cup S_j]$. Then the points in $S_{j+1}^* \cup \ldots \cup S_k^*$ lie outside the periphery of $H_j^*$ (j = 1, ..., k-1).

Proof: By definition of a (2, 1)-connected sequence, given any $w_i$ in S, there exists a path Q from $w_i$ to $w_n$ such that $int(Q) \subseteq \{w_{i+1}, \ldots, w_{n-1}\}$.

In particular, if $v \in S_{j+1} \cup \ldots \cup S_k$, there is a path Q from v to $w_n$ such that $int(Q) \subseteq S_{j+1} \cup \ldots \cup S_k$; hence Q is node-disjoint from $H_j$. Thus, since $H^*$ is planar, $v^*$ and $w_n^*$ must be on the same side of the periphery of $H_j^*$. (Recall that by Lemmas 3.3.1 and 3.3.2, the periphery

of $H_j^*$ is a closed Jordan curve.)  But by the hypothesis on $w_n^*$, $w_n^*$ must

lie outside the periphery of $H^*$.  Hence the lemma. ∎

Lemma 3.3.4   Given the hypotheses of Lemma 3.3.3, it follows that
$Base^*(S_{j+1}) \subseteq periphery(H_j^*)$.

Proof:   Since $Base(S_{j+1}) \subseteq S_1 \cup \ldots \cup S_j$ by Definition 2.3.3, each node

in $Base(S_{j+1})$ lies either within or on periphery $(H_j^*)$.  Moreover, by

Definition 2.3.3, each node in $Base(S_{j+1})$ is adjacent to a node in $S_{j+1}$.

However, by Lemma 3.3.3, $S_{j+1}^*$ lies outside periphery $(H_j^*)$.  Therefore,

by planarity of $H^*$, no node in $Base^*(S_{j+1})$ can lie inside periphery $(H_j^*)$.

Hence the lemma. ∎

Lemma 3.3.5   Suppose H is a nonseparable graph.  Suppose S is a (2, 1)-

connected sequence with segments $S_1, \ldots, S_k$.  Then $|Base(S_i)| \geq 2$ for

$i = 2, \ldots, k$.

Proof:   Let $\langle v_1, \ldots, v_t \rangle = S_i$.  By definition of segment (Definition 2.2.2),

$v_t$ is adjacent to two or more preceding nodes of S.  Hence by definition

of (2, 1)-connected sequence (Definition 2.2.1), the section graph of

$S_1 \cup \ldots \cup S_i$ (denoted $H[S_1 \cup \ldots \cup S_i]$) is nonseparable.  Now, since

$i \geq 2$, if $|Base(S_i)| = 0$, then $H[S_1 \cup \ldots \cup S_i]$ is disconnected, and if

$Base |(S_i)| = 1$, then $H[S_1 \cup \ldots \cup S_i]$ has Base $(S_i)$ as a 1-separating

set.  Hence, since $H[S_1 \cup \ldots \cup S_i]$ is nonseparable, it must be that

$|Base(S_i)| \geq 2$. ∎

Lemma 3.3.6   Suppose H is a nonseparable graph and S is a (2, 1)-

connected sequence for H, with segments $S_1, \ldots, S_k$.  Suppose

$S_j = \langle v_1, \ldots, v_t \rangle$ is a nonsingular segment of S.  Let

$\hat{S}_j = \langle anchor(S_j), v_1, \ldots, v_t \rangle$ (see Definition 2.3.2).

Then $\hat{S}_j$ is a path in H, and the interior nodes $v_1, \ldots, v_{t-1}$ of this path are not adjacent to any nodes of $(S_1 \cup \ldots \cup S_j) - \{v_t\}$.

Proof: By definition of segment (Definition 2.2.2), $S_j$ is a path, and by definition of anchor $(S_j)$ (Definition 2.3.2), $v_1$ is adjacent to anchor $(S_j)$. Hence $\hat{S}_j$ is a path. Moreover, if any of the nodes $v_1, \ldots, v_{t-1}$ were adjacent to a node in $S_1 \cup \ldots \cup S_j - \{v_t\}$, then $S_j$ would not be a segment. Hence the lemma. ∎

Lemmas 3.3.7a, b are the most important pair of lemmas in Section 3.1. Informally speaking, the sets $F(S_j)$ defined below correspond to the sets $F_j$ (j > 1) discussed in Chapter 2, Sections 2.1 and 3.1. Lemmas 3.3.7a, b show that the formal definition of "linear expansion" (and in particular, of "star") applies to $W = S_j$ (to within the set of "extra" edges mentioned in Chapter 2, Sections 2.1 and 3.1), thereby laying the groundwork for the proof of Theorem 3.3.12.

It is worthwhile to observe that, although "extra" edges can be present insofar as arbitrary (2, 1)-connected sequences are concerned, such edges never occur when we deal with the (2, 1)-connected sequences (or "primitive" (2, 1)-connected sequences) constructed in accord with Chapter 2. This absence of "extra" edges is guaranteed directly by the absence of internal chords in chains, and our construction of (2, 1)-connected sequences from chain sequences.

Lemma 3.3.7a Suppose H is a nonseparable graph and S is a (2, 1)-connected sequence on H. Let $S_1, \ldots, S_k$ denote the sequence of segments of S. Let $H_i = H[S_1 \cup \ldots \cup S_i]$ (i = 1, \ldots, k). Let $F(S_i)$ denote

the subgraph of H composed of $\{$edges with one node in $S_i$ and one node in $H_{i-1}\} \cup \{$edges with both nodes in $S_i\}$. Suppose $j > 1$. Suppose: $S_j$ is a singular segment; or $S_j = \langle v_1, \ldots, v_t \rangle$ is nonsingular and $v_t$ is not adjacent to any node in $\{$anchor$(S_j), v_1, \ldots, v_{t-2}\}$.

Then $F(S_j)$ is a star with center $v_t$. Also, $F(S_j)$ contains two or more nodes of $H_{j-1}$.

Lemma 3.3.7b  Suppose, in addition to the hypotheses of Lemma 3.3.6a, that H is planar and $H^*$ is a planar embedding of H with $w_n^*$ on the periphery of $H^*$ (where $w_n$ is last node of S). Let $H_{j-1}^*$ denote the restriction of $H^*$ to $H_{j-1}$. Let $P_{j-1}^*$ denote the periphery of $H_j^*$, and let $P_{j-1}$ denote the underlying circuit of $P_{j-1}^*$.

Then $G = H$, $C = P_{j-1}$, and $W = S_j = \langle v_1, \ldots, v_t \rangle$ satisfy the conditions specified in Definition 2.3.6: namely, $P_{j-1}$ is an elementary circuit in H, $S_j$ is a path in H, whose nodes belong to nodes(H)–nodes($P_{j-1}$), and $E_{P_{j-1}}(S_j) \overset{\text{def}}{=} \{$edges of H with one node in $S_j$ and one node in $P_{j-1}\} \cup \{$edges with both nodes in $S_j\}$

(see Definition 2.3.6) is a star with center $v_t$ and contains two or more nodes of $P_{j-1}$. Moreover, $E_{P_{j-1}}(S_j) = F(S_j)$.

Proof of 3.3.7a, b:  If $S_j$ is a singular segment (i.e., $t = 1$, and $S_j = \{v_1\} = \{v_t\}$), then $F(S_j)$ is a star with center $v_t$, since in this case $F(S_j) = \{$edges of H with one node $v_t$ and one node in $H_{j-1}\}$
    $= \{$edges $(v, v_t) \mid v \in$ Base$(S_j)\}$
by definition of Base$(S_j)$; also, $F(S_j)$ contains two or more nodes of $H_{j-1}$, since $|$Base$(S_j)| \geq 2$ by Lemma 3.3.5. So suppose $S_j$ is nonsingular, and $v_t$ is not adjacent to any node in $\{$anchor$(S_j), v_1, \ldots, v_t\}$. Then $F(S_j)$ consists of the path $\langle$anchor$(S_j), v_1, \ldots, v_t \rangle$ plus $\{(v, v_t) \mid v \in$ Base$(S_j)\}$

3-49

3-49

($\langle \text{anehor}(S_j), v_1, \ldots, v_t \rangle$ is a path by Lemma 3.3.6), so $F(S_j)$ is a star with center $v_t$; also, $F(S_j)$ contains two or more nodes of $H_{j-1}$, sinee $|\text{Base}(S_j)| \geq 2$ by Lemma 3.3.5.

To prove part (b), we observe first that $P_j$ is an elementary eireuit in H by Lemma 3.3.1 and Lemma 3.3.2; also, $S_j$ is a path in H by Lemma 3.3.6, and $\text{nodes}(S_j) \subseteq \text{nodes}(H) - \text{nodes}(P_{j-1})$ sinee $\text{nodes}(P_j) \subseteq S_1 \cup \ldots \cup S_{j-1}$. To prove that $E_{P_{j-1}}(S_j)$ is a star with center $v_t$, it suffiees by part (a) to show that $E_{P_{j-1}}(S_j) = F(S_j)$. By definition, $E_{P_{j-1}}(S_j)$ is a subgraph of $F(S_j)$; to show that the two are identical, it is sufficient to show that if $e = (w_1, w_2)$ is an edge of H with one node $w_1$ in $S_j$ and the other node $w_2$ in $H_{j-1}$, then $w_2$ belongs to $P_{j-1}$ — and this assertion is valid beeause $w_1 \in S_j$, $w_2 \in H_{j-1}$ implies $w_2 \in \text{Base}(S_j)$ by definition, and $\text{Base}(S_j) \subseteq \text{nodes}(P_{j-1})$ by Lemma 3.3.4. Since $E_{P_{j-1}}(S_j) = F(S_j)$, it follows at onee from part (a) and the fore-going remarks on $e = (w_1, w_2)$ that $E_{P_{j-1}}(S_j)$ eontains two or more nodes of $P_{j-1}$. ∎

As noted earlier, Lemmas 3.3.8 through 3.3.11 establish technical results needed in the proof of Theorem 3.3.12. Lemma 3.3.8 proves that the set $F_1$ (defined in Chapter 2, Section 2.1) consists of an elementary circuit plus a set of "extra" edges, as claimed in Chapter 2, Section 2.1. Lemma 3.3.9 proves the existenee of a planar embedding with a speeialty selected edge on its periphery. Lemma 3.3.10 states that the underlying eireuit of the periphery of a linear expansion is unique. Lemma 3.3.11 shows that "extra" edges always lie <u>within</u> peripheries of interest to us; this property is used in Theorem 3.3.12 to show that "extra" edges cause no problems for proving the validity of the Expansion Algorithm.

Lemma 3.3.8   Given a nonseparable graph $H = (V, E)$ and a $(2, 1)-$
connected sequence S with segments $S_1, \ldots, S_k$. Let $C_1 = \langle v_1, \ldots, v_t, v_1 \rangle$,
where $\langle v_1, \ldots, v_t = S_1$. Then $C_1$ is an elementary circuit in H. Let
$E_1 = \{ e \in E | e = (v_t, v_i) \text{ and } i \in \{2, \ldots, t-2\} \}$. Then $C_1 \cup E_1$ is the section
graph of $S_1$.

Proof:   By definition of a segment (Definition 2.2.2), $v_t$ is the first node
of S which is adjacent to a preceding node, and the section graph of S is
nonseparable; hence $v_t$ is adjacent to $v_1$. Thus $C_1$ is a circuit. More-
over, $C_1$ is an elementary circuit, since $v_1, \ldots, v_t$ are all distinct.

By definition, $C_1 \cup E_1$ is a subgraph of the section graph of $S_1$
(denoted $H[S_1]$). Moreover, $C_1$ has no chords other than those in $E_1$,
since $v_t$ is the <u>first</u> node of S that is adjacent to a preceding node in S.
Hence $C_1 \cup E_1 = H[S_1]$. ∎

Lemma 3.3.9   Suppose H is a planar, 3-connected graph and
$S = \langle w_1, \ldots, w_n \rangle$ is a primitive $(2, 1)-$connected sequence for H.
(a)   Then there exists a planar realization $H^*$ of H in which both $w_n$ and
$e_0^* = (w_1^*, w_2^*)$ lie on the periphery of $H^*$.

(b)   Let $S_1, \ldots, S_k$ denote the segments of S. Let
$H_j^* = H^*[S_1 \cup \ldots \cup S_j]$ $(1 \leq j \leq k)$. Then $e_0^*$ lies on the periphery of
$H_j^*$ $(1 \leq j \leq k)$.

Proof:   Since S is a primitive $(2, 1)-$connected sequence, there exists a
primitive circuit C (see Definition 2.2.13) in H, containing both $w_n$ and
the edge $(w_1, w_2)$. Let $K^*$ denote any planar embedding of H. By
[MacL37b], (page 466, Theorem 6), C must be a boundary of a face of $K^*$.
By drawing $K^*$ on the sphere and employing stereographic projection

3-51

(see [Ore67]) we may transform $K^*$ into a planar embedding $H^*$ having C as the boundary of the infinite face, i.e., as the periphery. Hence the first part of the lemma.

The second part of the lemma follows at once, since if $e_0^*$ lies inside the periphery of any $H_j^*$, it cannot lie on the periphery of $H^*$. ∎

<u>Lemma 3.3.10</u>  Suppose $C, G, C^*, e, w$ satisfy the conditions of Definition 2.3.6. That is, suppose C is an elementary circuit in the graph G, $C^*$ is a planar embedding of C, and e is an edge of C; suppose $W = \langle w_1, \ldots, w_t \rangle$ is a path in G, whose nodes belong to nodes(G)-nodes(C); let $E_C(W) = \{$edges of G with one node in W and one node in C$\} \cup$
$\qquad \{$edges with both nodes in W$\}$,
and suppose $E_C(W)$ is a star with center $w_t$, and $E_C(W)$ contains two or more nodes of C.

Suppose $D_1^*, D_2^*$ are two linear expansions of $C^*$ (based on e) relative to W. Then boundary $(D_1^*)$ = boundary $(D_2^*)$, where by "boundary" we mean the underlying graph of the periphery.

<u>Proof:</u>  Since $E_C(W)$ is a star by hypothesis, the lemma follows directly from the definition of linear expansion and fundamental topological theorems on the plane — notably the Jordan Curve Theorem — which are assumed at the outset. (In particular, the requirement that edge $e^*$ lie on the periphery of the linear expansion uniquely determines that periphery.) ∎

<u>Lemma 3.3.11</u>  Suppose $H, S, H^*, H_j^*$ satisfy the hypotheses of Lemma 3.3.3. Suppose $S_j = \langle v_1, \ldots, v_t \rangle$ is a nonsingular segment of S and that there exists at least one edge in H with one endpoint $v_t$ and the other

endpoint in $\{\text{anchor}(S_j), v_1, \ldots, v_{t-2}\}$. Let $\overline{E}$ = the set of all such edges; let $\overline{E}^*$ = the set of edges of $H^*$ corresponding to $\overline{E}$.

If H is 3-connected, then all edges of $\overline{E}^*$ lie inside the periphery of $H_j^*$.

<u>Proof</u>: By definition of $S_j$, $v_{t-1}$ is adjacent to no nodes of $H_j$ other than $v_{t-2}$ and $v_t$ (where, if $t = 2$, we define $v_{t-2} = \text{anchor}(S_j)$). Thus $v_{t-1}$ is of degree 2 relative to $H_j$. Moreover, no node in $S_{j+1} \cup \ldots \cup S_k$ is adjacent to any node within the periphery of $H_j^*$, since by Lemma 3.3.3, all such nodes lie outside the periphery of $H_j^*$.

However, if any edge of $\overline{E}^*$ lies on the periphery of $H_j^*$, then $v_{t-1}$ lies within that periphery and hence is not adjacent to any node in $S_{j+1} \cup \ldots \cup S_k$; therefore, by the preceding paragraph, $v_{t-1}$ is a node of degree 2 relative to H, contradicting the 3-connectedness of H. Thus all edges of $\overline{E}^*$ must lie inside the periphery of $H_j^*$. ∎

3-53

3.2 Expansion Algorithm/Linear Version

The following theorem establishes the validity of the Expansion Algorithm/Linear Version.

<u>Theorem 3.3.12</u>  Suppose G is a 3-connected graph and S is any primitive (2, 1)-connected sequence for G.  Then G is planar if and only if the Expansion Algorithm/Linear Version (EA/L) (Chapter 2, Section 3.2), applied to S, produces a planar embedding of G.

<u>Proof:</u>  It suffices to assume that G is planar, and prove that EA/L produces a planar embedding of G.  Suppose $S = \langle w_1, \ldots, w_n \rangle$.  Let $S_1, \ldots, S_k$ denote the sequence of segments of S.  Let $H^*$ be a planar embedding of G in which $w_n^*$ and edge $e_0^* = (w_1^*, w_2^*)$ lie on the periphery. By Lemma 3.3.9, such an embedding exists.  Let $H_j^*$ denote the restriction of $H^*$ to the section graph of $S_1 \cup \ldots \cup S_j$.

By Lemma 3.3.1, $H_j$ is a nonseparable graph.  Let $P_j^*$ be the periphery of $H_j^*$; let $P_j$ be the underlying graph of $P_j^*$.  By Lemma 3.3.2, $P_j$ is an elementary circuit.  By Lemma 3.3.3, the nodes of $S_{j+1}^*$ lie outside $P_j^*$.  Thus, by planarity of $H^*$ and Lemma 3.3.4:

(3.3.13)  . . . .  $\quad$ $\text{Base}(S_{j+1}) \subseteq \text{nodes}(P_j)$ .

Suppose $G_j^*$ has been constructed in accord with EA/L.  By construction, $G_j^*$ is a planar embedding of the section graph of $S_1 \cup \ldots \cup S_j$. (Let us verify this assertion by finite induction.  Indeed, $G_1^*$ is a planar embedding of $C_1 \cup E_1$ (as defined in EA/L), and by Lemma 3.3.8, $C_1 \cup E_1$ = section graph of $S_1$; furthermore, (since $\text{Base}(S_j) \subseteq \text{nodes}(C_{j-1})$ by assumption on $G_j^*$) $G_{j-1} \cup E_{C_{j-1}}(S_j)$ = section graph of $S_1 \cup \ldots \cup S_j$, and by construction, $G_j^*$ is a planar embedding of $G_{j-1} \cup E_{C_{j-1}}(S_j)$

(here $G_{j-1}$ denotes the underlying graph of $G_{j-1}^{*}$).)

Hence, in order to prove the theorem, it suffices (in view of (3.3.13)) to show that $C_j = P_j$, given that $C_j$ has been constructed in accord with EA/L. That is, it suffices to prove:

Lemma 3.3.14   Suppose $G_j^{*}, C_j^{*}, C_j$ have been constructed in accord with EA/L, for some $j \in \{1, \ldots, k\}$. Then $C_j = P_j$, where $P_j$ is defined as above.

Proof:   We proceed by induction on j. $C_1 = P_1$ because $C_1 = \langle v_1, \ldots, v_t, v_1 \rangle$ by construction.

So suppose the lemma is satisfied for $j = i-1 \geq 1$. We wish to show it must therefore be satisfied for $j=i$. We may assume, then, that $G_i^{*}, C_i^{*}, C_i$ have been constructed in accord with EA/L and that $C_{i-1} = P_{i-1}$.

Case 1   Suppose $S_i$ is singular, or else $S_i = \langle v_1, \ldots, v_t \rangle$ is non-singular, and $v_t$ is not adjacent to any node in $\{anchor(S_i), v_1, \ldots, v_{t-2}\}$. By Lemma 3.3.7b, $W = S_i$, $C^{*} = P_{i-1}^{*}$ satisfy the conditions of Definition 2.3.6 (Chapter 2). By Lemma 3.3.3, the points corresponding to $W = S_i$ lie outside $P_{i-1}^{*}$; also, by definition of $H^{*}$, the line corresponding to $e_0 = (w_1, w_2)$ lies on $P_i^{*}$. Thus, allowing a slight abuse of language, the embedding $H_i^{*}$ is a linear expansion of $H_{i-1}^{*}$ (based on $e_0$) relative to $S_i$. Hence, since $C_{i-1} = P_{i-1}$, Lemma 3.3.10 implies that $C_i = P_i$.

Case 2   Suppose the hypothesis of Case 1 is not satisfied; i.e., suppose $S_i = \langle v_1, \ldots, v_t \rangle$ is a nonsingular segment and suppose there exist edges with one endpoint $= v_t$ and one endpoint in $\{anchor(S_i), v_1, \ldots, v_{t-2}\}$. Let $\bar{E}$ denote the set of all such edges.

By Lemma 3.3.11, the edges of $H_i^*$ corresponding to $\overline{E}$ lie inside $P_i^* = $ periphery of $H_i^*$. Also, by the construction L.2.2 in EA/L, the edges of $G_i^*$ corresponding to $\overline{E}$ lie inside $C_i^*$ (moreover, it must be mentioned that the ability to embed $\overline{E}$ in a planar fashion, inside $C_i^*$ is guaranteed by Lemma 3.3.6). Hence, the existence of the extra edges $\overline{E}$ has no effect on the peripheries, so that the same reasoning as that in Case 1 implies that $C_i = P_i$.

Hence the lemma.

As noted above, Lemma 3.3.14 completes the proof of Theorem 3.3.12. ∎

## 3.3  Expansion Algorithm/Peripheral Version

The validity of the Expansion Algorithm/Peripheral Version (Chapter 2, Section 3.3) will be shown to follow directly from the succeeding lemmas. These lemmas will be used to show that, given a nonseparable, planar graph G, and a $(2,1)$-connected sequence for G with segments $S_1, \ldots, S_k$, the Expansion Algorithm/Peripheral Version does not stop until $G_k^*$ has been constructed. It is then shown in Theorem 3.3.17 that $G_k^*$ is a planar embedding of G.

**Lemma 3.3.15** Suppose H is a nonseparable, planar graph. Suppose $S = \langle w_1, \ldots, w_n \rangle$ is a $(2\text{-}1)$-connected sequence for G, with segments $S_1, \ldots, S_k$. Suppose $H^*$ is a planar embedding of H with $w_n$ on its periphery.

Suppose C is any elementary circuit in $H[S_1 \cup \ldots \cup S_i]$, $i \in \{1, \ldots, k\text{-}1\}$, containing all deficient nodes of $S_1 \cup \ldots \cup S_i$ (see Definition 2.3.9). Then the periphery of $H^*[C \cup S_{i+1}]$ contains all deficient nodes of $S_1 \cup \ldots \cup S_{i+1}$.

**Proof:** Since C contains all deficient nodes of $S_1 \cup \ldots \cup S_i$, it follows that $C \cup S_{i+1}$ contains all deficient nodes of $S_1 \cup \ldots \cup S_{i+1}$. Let $H_{i+1}^*$ denote $H^*[S_1 \cup \ldots \cup S_{i+1}]$. By way of contradiction, suppose that there is a deficient node v of $S_1 \cup \ldots \cup S_{i+1}$ which does not lie on the periphery of $H^*[C \cup S_{i+1}]$; then since $v \in \text{nodes}(C \cup S_{i+1})$, v must lie inside this periphery, and hence inside the periphery of $H_{i+1}^*$ (since nodes $(C \cup S_{i+1}) \subseteq S_1 \cup \ldots \cup S_{i+1}$). Moreover, since v is a deficient node of $S_1 \cup \ldots \cup S_{i+1}$, it is adjacent to at least one node in $S_{i+2} \cup \ldots \cup S_k$ (by definition). Yet by Lemma 3.3.3, all nodes of $S_{i+2} \cup \ldots \cup S_k$ lie

outside the periphery of $H^*_{i+1}$. Thus we contradict the planarity of $H^*$.

Hence the lemma. ∎

Lemma 3.3.16   Suppose H is a nonseparable, planar graph. Suppose $S = \langle w_1, \ldots, w_n \rangle$ is a (2, 1)-connected sequence for H, with segments $S_1, \ldots, S_k$.

Suppose C is any elementary circuit in $H[S_1 \cup \ldots \cup S_i]$, containing all deficient nodes of $S_1 \cup \ldots \cup S_i$, $i \in \{1, \ldots, k-1\}$. (In particular, therefore, $\text{Base}(S_{i+1}) \subseteq \text{nodes}(C)$ by definition, and $|\text{Base}(S_{i+1})| \geq 2$ by Lemma 3.3.5.) Then there exists a sector $P = \langle x_1, \ldots, x_q \rangle$ of C, determined by $\text{Base}(S_{i+1})$ (see Definition 2.3.8) such that

$$P \cup (_{x_1}S_{i+1}) \cup (_{x_q}S_{i+1})$$

contains all deficient nodes of $S_1 \cup \ldots \cup S_{i+1}$. (See Definition 2.3.7 for meaning of "$_x S_j$".)

Proof:   Let $H^*$ be a planar embedding of H, having $w_n$ on its periphery. By Lemma 3.3.15, the periphery of $H^*[C \cup S_{i+1}]$ contains all deficient nodes of $S_1 \cup \ldots \cup S_{i+1}$.

First we show that $S^*_{i+1}$ lies outside $H^*[C]$. Let $H^*_i$ denote $H^*[S_1 \cup \ldots \cup S_i]$. By Lemma 3.3.3, $S^*_{i+1}$ lies outside the periphery of $H^*_i$. Hence, since C lies in $H[S_1 \cup \ldots \cup S_i]$, $S^*_{i+1}$ lies outside $H^*[C]$.

Let $\langle v_1, \ldots, v_t \rangle = S_{i+1}$. Suppose that $v_t$ is not adjacent to any node in $\{\text{anchor}(S_i), v_1, \ldots, v_{t-2}\}$. Then by basic topological properties of the plane, and the fact that $S^*_{i+1}$ lies outside $H^*[C]$, it follows that the periphery of $H^*[C \cup S_{i+1}]$ is of the form

$$P \cup (_{x_1}S_{i+1}) \cup (_{x_q}S_{i+1}),$$

where $P = \langle x_1, \ldots, x_q \rangle$ is a sector of C determined by $Base(S_{i+1})$ (observe that the set of edges $E_C(S_{i+1})$ (see Definition 2.3.6) is a star, by definition of <u>segment</u> and the above assumption on $v_t$). Hence, with the above assumption on $v_t$, the lemma is satisfied, since the periphery of $H^*[C \cup S_{i+1}]$ contains all deficient nodes of $S_1 \cup \ldots \cup S_{i+1}$, as noted above.

On the other hand, suppose that $v_t$ is adjacent to one or more nodes in $\{anchor(S_i), v_1, \ldots, v_{t-2}\}$. The lemma still holds in this case, since the nodes of the periphery of $H^*[C \cup S_{i+1}]$ must be a <u>subset</u> of the nodes in $P \cup (_{x_1} S_{i+1}) \cup (_{x_q} S_{i+1})$, for some sector $P = \langle x_1, \ldots, x_q \rangle$ of C. Hence the lemma. ∎

Theorem 3.3.17   Suppose G is a nonseparable graph and S is any $(2, 1)$-connected sequence for G. Then G is planar if and only if the Expansion Algorithm/Peripheral Version (EA/P) (Chapter 2, Section 3.3), applied to S, produces a planar embedding of G.

Proof:   Let $S_1, \ldots, S_k$ denote the sequence of segments of S. Suppose that $G_j^*, C_j^*, C_j$ $(j \in \{1, \ldots, k-1\})$ have been constructed in accord with EA/P. First we will show that $C_j$ is necessarily an elementary circuit containing all deficient nodes of $S_1 \cup \ldots \cup S_j$. Then we will show that the theorem follows from Lemma 3.3.16.

We proceed by induction on j. Suppose j = 1. By construction, $C_1 = \langle v_1, \ldots, v_t, v_1 \rangle$, where $\langle v_1, \ldots, v_t \rangle = S_1$. Then $C_1$ is an elementary circuit, by Lemma 3.3.8; moreover, $C_1$ contains <u>all</u> nodes of $S_1$. Now suppose $G_j^*, C_j^*, C_j$ have been constructed in accord with EA/P,

from $G_{j-1}^*$, $C_{j-1}^*$, $C_{j-1}$, where $j \in \{2, \ldots, k-1\}$, and suppose the induction hypothesis on j-1 is satisfied: i.e., $C_{j-1}$ is an elementary circuit containing all deficient nodes of $S_1 \cup \ldots \cup S_{j-1}$. Since $G_j^*$ has been constructed in accord with EA/P, it follows that there exists a sector $P = \langle x_1, \ldots, x_q \rangle$ of $C_{j-1}$, determined by Base($S_j$), such that $P \cup (_{x_1} S_j) \cup (_{x_q} S_j)$ contains all deficient nodes of $S_1 \cup \ldots \cup S_j$. By definition of EA/P, $G_j^*$ is constructed so that the periphery of $G_j^*$ (namely, $C_j$) equals $P \cup (_{x_1} S_j) \cup (_{x_q} S_j)$: indeed, $G_j^*$ is constructed by linear expansion, based on an edge in such a P, relative to $S_j$ (in this discussion we may ignore the possible presence of "extra edges" adjacent to the last node of $S_j$, since such edges are placed within the periphery of $G_j^*$), and by basic topological properties of the plane, the periphery of such a linear expansion consists of the nodes and edges in $P \cup (_{x_1} S_j) \cup (_{x_q} S_j)$. This completes the induction.

In particular, it follows that Base($S_{j+1}$) $\subseteq$ nodes($C_j$), as claimed in Chapter 2, Section 3.3 (inasmuch as any node in Base($S_{j+1}$) is, by definition, a deficient node of $S_1 \cup \ldots \cup S_j$).

We will now use the foregoing property of $C_j$ with regard to deficient nodes (in conjunction with Lemma 3.3.16) to establish the present theorem. It suffices to assume that G is planar, and show that EA/P produces a planar embedding of G. So suppose G is planar. By the foregoing property of $C_j$, Lemma 3.3.16 applies ($C_j$ is an elementary circuit by Lemma 3.3.2, since it is the periphery of $G_j^*$, $G_j^*$ is a planar embedding of $G[S_1 \cup \ldots \cup S_j]$, and $G[S_1 \cup \ldots \cup S_j]$ is nonseparable by Lemma 3.3.1); hence the algorithm EA/P does not stop until $G_k^*$ has been constructed.

Thus it suffices to show that $G_k^*$ is a planar embedding of G. By construction, $G_j^*$ is planar, and is an embedding of the section graph of $S_1 \cup \ldots \cup S_j$ (j = 1, . . . , k). (Indeed, $G_1^*$ is a planar embedding of $C_1 \cup E_1$ (as defined in step (L.1), Chapter 2, Section 3.2), and by Lemma 3.3.8, $C_1 \cup E_1$ is the section graph of $S_1$; furthermore, by EA/P, $G_j^*$ is a planar embedding of $G_{j-1} \cup E_{C_{j-1}}(S_j)$ (see Definition 2.3.6), where $G_{j-1}$ denotes the underlying graph of $G_{j-1}^*$ — and since $Base(S_j) \subseteq nodes(C_{j-1})$, this union is in fact the section graph of $S_1 \cup \ldots \cup S_j$.) Thus $G_k^*$ is a planar embedding of the section graph of $S_1 \cup \ldots \cup S_k$, that is, of G itself.

This completes the proof of Theorem 3.3.17. ∎

Chapter 4

## CONCLUSION

## Section 1. COMPUTATIONAL COMPLEXITY

It will be demonstrated in this section that the planarity-testing algorithm described in this thesis is $O(n^2)$, where n = number of nodes in the graph.

In the following theorem and its proof, the expression "algorithm X is $O(m)$" means that X can be realized by a Turing machine which performs a bounded number of "steps," where the bound is $k \cdot m$ (k independent of m); the term "step" refers to the process of executing a machine instruction. We assume that the reader is familiar with the Turing machine concept; in our proofs, we feel it is unnecessary to reduce our arguments to the Turing machine level. A similar comment applies to the use of "$O(m^2)$."

Also, we should mention that, although use of the expression "choose any ..." would appear to make the algorithm nondeterministic, this appearance is purely artificial, since only finite structures are utilized.

We should stress that the following theorem is not intended to establish the best lower bound. It has been included only in order to point out that the efficiency of the algorithms developed here is <u>at least</u> as good as $O(n^2)$. The main objective of our work has been to clarify and establish the validity of these algorithms.

4-2

Considerably more work can be done on the question of the best lower bound. We might mention at least two points at which the effects of clever programming might be explored. First, in step (T.3.1) if T is the subgraph from which path Q is extracted, then under suitable circumstances T might be replaced, during the next execution of (T.2), by the section graph of nodes(t)-nodes(Q): in this way, the effort of finding a "new" T might be reduced. Second, in step (T.4.7), subgraph T' is found by testing several possible candidates; these other sub-graphs might be saved for future use, to avoid wasting steps in gener-ating them again. Additional sources of increased efficiency might be discovered by considering, for each occurrence of the expression "choose any ...," the possibility of an especially judicious choice.

Theorem 4.1.1  Suppose G is a 3-connected graph. The planarity-testing algorithm composed of (T.1)-(T.5) and (P.1)-(P.3), followed by (L.1)-(L.2.2) (Chapter 2, Sections 2 and 3) is $O(n^2)$, where n = the number of nodes in G.

Proof:  It suffices to show that the test is $O(m^2)$, where m = the number of edges in G. Indeed, if G is planar, then m ≤ 3n - 6 (see [Liu68], p. 210). Hence, we begin by counting the number of edges of G, discontinuing the count if 3n-6 is exceeded (in which case G is nonplanar). Hence we may assume that m ≤ 3n-6, and it suffices to show that the planarity test is $O(m^2)$.

First, we will show that algorithm (T.1)-(T.5) is $O(m^2)$. Let us begin with (T.1)-(T.4.8), the part of the algorithm which generates a chain sequence. Algorithm (T.1)-(T.4.8) consists of (T.1), executed once,

followed by (T.2)-(T.4.8), executed iteratively. As proven in Theorem 3.2.22 (Chapter 3), each iteration of (T.2)-(T.4.8) strictly augments the number of nodes in the pre-chain sequence r, being constructed. Hence at most n such iterations can occur, so it suffices to show each iteration is O(m). Moreover, within each iteration of (T.2)-(T.4.8), each step (T.2) through (T.4.8) is executed either zero times, once, or (if a reduction to a previous case occurs) (see Theorem 3.2.22) twice. Hence, it suffices to show that each of the steps (T.1) and T.2) through (T.4.8), is O(m).

Step (T.1) is O(m), since it suffices to apply Lee's algorithm once, and Lee's algorithm is O(m) (see [Lee61]). Step (T.2) is O(m), since the construction of subgraph T can be accomplished by Lee's algorithm. Step (T.3) is O(m), since $|A(T)| \leq n$. Step (T.3.1) is O(m), since Q may be constructed by Lee's algorithm (and insertion of Q into r requires no more than O(n) steps; the same is true of insertion of paths in later steps, so this point will not be repeated below). Step (T.3.2) is a go to. Step (T.3.2) is O(m), since s(R) can be constructed in < 2m steps by marking each node in int(S), (S > R) and then checking each node in int(R) for adjacency with a marked node. By the same argument, construction of p(R) is O(m). This point will not be continually repeated below. Steps (T.3.4) and (T.3.5) are each O(m), since checking of adjacency is an O(m) process. Step (T.3.6) is a go to.

Step (T.4) is an O(m) process, since determining whether or not two sets of nodes have empty intersections is an O(m) process. Step (T.4.1) is O(m), since Q may be constructed by Lee's algorithm. Step (T.4.2) is a go to. Step (T.4.3) is O(m), since finding a node in the nonempty intersection of two sets of nodes can be done in 2n steps.

Steps (T.4.4) and (T.4.5) are each $O(m)$, for the same reason as given for (T.3.4), (T.3.5). Step (T.4.6) is $O(m)$, since determining the difference of two sets of nodes is an $O(m)$ process.

Step (T.4.7) is $O(m)$, for the following reasons. First, construction of $s(R)$ and $p(R)$ is $O(m)$, as noted above; hence construction of $K$ is $O(m)$. By Lemma 3.2.14, construction of $T'$ satisfying (a)-(e) is equivalent to construction of a bridge $B'$ satisfying $\overline{A}(B') \cap \text{int}(K) \neq \phi$ and $\overline{A}(B') \not\subseteq \text{nodes}(K)$ (by Lemma 3.2.20, such a bridge exists). However, by giving nodes in $\text{int}(K)$ a special mark, the search for such a $B'$ can be accomplished simply by generating all the bridges $B$, and checking, for each, whether $\overline{A}(B)$ contains both a marked and an unmarked node. Since generation of all the bridges is an $O(m)$ process, step (T.4.7) is $O(m)$. Step (T.4.8) is $O(m)$, since it contains no operations not already touched upon. Step (T.5) is $O(m)$, since it involves merely concatenation of paths that have been generated already.

Hence, algorithm (T.1)-(T.5) is indeed $O(m^2)$. Also, algorithm (P.1)-(P.3) is $O(m^2)$, since it involves only two applications of algorithm (T.1)-(T.5). Furthermore, algorithm (L.1)-(L.2.2) is $O(m^2)$, since it involves at most $k < n$ iterations ($k$ = number of segments of $S$) and each iteration is $O(n)$ (indeed, checking if "$\text{Base}(S_{j+1}) \not\subseteq \text{nodes}(C_j)$" is an $O(n)$ operation).

Hence the theorem. ∎

Theorem 4.1.2   Suppose $G$ is a 3-connected graph. The planarity-testing algorithm composed of (T.1)-(T.5), followed by the Expansion Algorithm/Peripheral Version (Chapter 2, Sections 2 and 3) is $O(n^2)$, where $n$ = the number of nodes in $G$.

Proof: By the preceding theorem, it suffices to observe that the test contained in paragraph 2.3.11 (see Chapter 2, subsection 3.3) is $O(n)$, and is performed no more than n times. ∎

It should be noted that all the foregoing algorithms require storage space which is no more than linearly related to m; namely, the only data structures needed are: (1) the adjacency relation defining the graph (which can be stored in space proportional to m by list structure techniques — see, e.g. [Shir 69]); (2) a list containing a pre-chain sequence undergoing augmentation; (3) a list containing a (primitive) (2, 1)-connected sequence; and (4) a list or ring representing the periphery of $G_j^*$ (these last three lists can be stored in space proportional to m, by their very definition).

## Section 2.  FUTURE RESEARCH AND APPLICATIONS

The existence of $(2, 1)$-connected sequences for arbitrary
(planar or nonplanar) 3-connected graphs leads us to believe that
such sequences may be relevant to the study of geometric embed-
ding of nonplanar graphs.  For example, a $(2, 1)$-connected sequence
for the graph $(K_{3,3})$ in Figure 4.1 is $S = \langle A, X, B, Z, Y, C \rangle$.
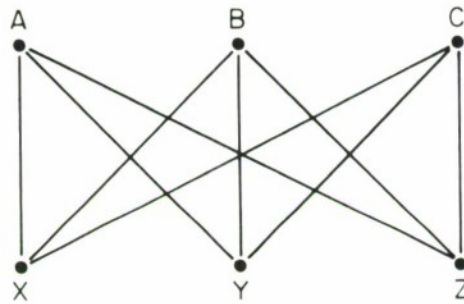


Figure 4.1    Graph $K_{3,3}$

This sequence generates an uncomplicated 2-planar layout of $K_{3,3}$
(see Figure 4.2) by a straightforward extension of the "expansion
algorithm" technique:  namely, when an edge cannot be included
without overlap of edges, attempt to place this "nonplanar" edge
(e.g., the dotted edge in Figure 4.2) on a second plane.

Figure 4.2   Two-Planar Embedding of $K_{3,3}$
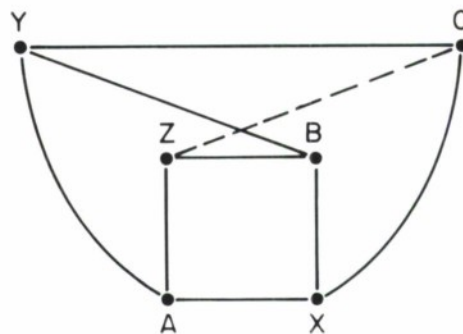
Experiments with much larger graphs have suggested that this technique may be of practical and theoretical interest.

Furthermore, (2, 1)-connected sequences might be studied for their possible application to display techniques for complicated networks, where an important objective is psychological "clarity," or simplicity in generating the geometric description of the display.

## Section 3. SUMMARY

The existence and constructability of a new graph-theoretic structure — the "(2, 1)-connected sequence" — has been proven for arbitrary (planar or nonplanar) 3-connected graphs. It has been proven that such sequences can be used to test graphs for planarity, and to construct planar embeddings of planar graphs, via a very simple technique called the "expansion algorithm." A distinguishing feature of this technique is that the embedding is constructed in stages, from the interior outward; at each stage, no reference need be made to the nodes or edges within the periphery of the embedding constructed in the preceding stages.

All algorithms have been proven to operate within time $O(n^2)$, where n is the number of nodes in the graph being tested for planarity.

The most striking feature of this approach to planarity is that essentially all the "work" of the test consists of generating a (2, 1)-connected sequence, whose definition is expressed solely in terms of connectivity. Thus, such sequences establish a new and close connection between the study of connectivity and the study of embedding. In particular, the existence of (2, 1)-connected sequences for nonplanar graphs opens new possibilities for the exploration of geometric embedding problems of nonplanar graphs.

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13. ABSTRACT

This thesis introduces a new graph-theoretic structure – the $(2,1)$-connected sequence – with direct applicability to the embedding of both planar and nonplanar graphs. It is proven that: (1) the nodes of a graph can be ordered so as to form a $(2,1)$-connected sequence, regardless of whether the graph is planar or nonplanar, and (2) such a sequence yields a new and exceptionally simple technique for planarity testing and embedding. All algorithms are proven to operate within a time bound proportional to the square of the number of nodes or edges in the graph.

14. KEY WORDS

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|---|---|
| graph theory | integrated circuit layout |
| nonplanar graphs | combinatorial ordering |
| planar graphs | geometric embedding of graphs |
| embedding algorithms | graph-theoretic structure |